QUANTUM GENERALIZED KAC–MOODY ALGEBRAS VIA HALL ALGEBRAS OF COMPLEXES

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ABSTRACT. We establish an embedding of the quantum enveloping algebra of a symmetric generalized Kac–Moody algebra into a localized Hall algebra of $\mathbb{Z}_2$-graded complexes of representations of a quiver with (possible) loops. To overcome difficulties resulting from the existence of infinite dimensional projective objects, we consider the category of finitely-presented representations and the category of $\mathbb{Z}_2$-graded complexes of projectives with finite homology.

1. INTRODUCTION

Let $\mathcal{A}$ be an abelian category such that the sets $\mathrm{Hom}(A, B)$ and $\mathrm{Ext}^1(A, B)$ are both finite for all $A, B \in \mathcal{A}$. The *Hall algebra* of $\mathcal{A}$ is defined to be the $\mathbb{C}$-vector space with basis elements indexed by isomorphism classes in $\mathcal{A}$ and with associative multiplication which encodes information about extensions of objects. Typical examples of such abelian categories arise as the category $\mathrm{rep}_{\Bbbk}(\mathcal{Q})$ of finite-dimensional representations of an acyclic quiver $\mathcal{Q}$ over a finite field $\Bbbk := \mathbb{F}_q$. This category became a focal point of intensive research when C. Ringel [20] realized one half of a quantum group via a twisted Hall algebra of the category. This twisted Hall algebra is usually called the *Ringel–Hall* algebra. The construction was further generalized by J. A. Green [9] to one half of the quantum group of an arbitrary Kac–Moody algebra.

Even though there is a construction, called *Drinfeld double*, which glues together two copies of one-half quantum group to obtain the whole quantum group, it is desirable to have an explicit realization of the whole quantum group in terms of a Hall algebra. Among various attempts, the idea of using a category of $\mathbb{Z}_2$-graded complexes was suggested by the works of M. Kapranov [15], L. Peng and J. Xiao [17, 18]. In his seminal work [4], T. Bridgeland successfully utilized this idea to achieve a Hall algebra realization of the whole quantum group. More precisely, given a Kac–Moody algebra $\mathfrak{g}$, he took the category $\mathrm{rep}_{\Bbbk}(\mathcal{Q})$ of finite dimensional representations of an acyclic quiver $\mathcal{Q}$ associated with $\mathfrak{g}$, and considered the full subcategory $\mathcal{P}$ of projective objects in $\mathrm{rep}_{\Bbbk}(\mathcal{Q})$. By studying the category $\mathcal{C}(\mathcal{P})$ of $\mathbb{Z}_2$-graded complexes in $\mathcal{P}$, he showed that the whole quantum group is embedded into the reduced localization of a twisted Hall algebra of $\mathcal{C}(\mathcal{P})$.

The purpose of this paper is to extend Bridgeland's construction to *generalized Kac–Moody algebras*. These algebras were introduced by R. Borcherds [2] around 1988. He used a generalized Kac–Moody algebra, called the *Monster Lie algebra*, to prove the celebrated Moonshine Conjecture [3]. Since then, many of the constructions in the theory of Kac–Moody algebras have been extended to generalized Kac–Moody algebras. In particular, the quantum group of a generalized Kac–Moody

[†]This paper was supported by the National Research Foundation of Korea (NRF) funded by the Ministry of Science (NRF-2017R1C1B5018384).

[*]This work was partially supported by a grant from the Simons Foundation (#318706).

algebra was defined by Kang [12], and one half of the quantum group was realized via a Hall algebra by Kang and Schiffmann [13], following Ringel–Green's construction.

The main difference from the usual Kac–Moody case is that the quiver $\mathcal{Q}$ may have loops in order to account for imaginary simple roots. A natural question arises:

> *Is it possible to realize the whole quantum group of a generalized Kac–Moody algebra in terms of a Hall algebra of $\mathbb{Z}_2$-graded complexes?*

A straightforward approach would run into an obstacle. Namely, when there is a loop, a projective object may well be infinite dimensional, and the Hall product would not be defined.

In this paper, we show that this difficulty can be overcome by considering the category $\mathcal{R}$ of finitely-presented representations of a locally finite quiver $\mathcal{Q}$, possibly with loops, and the category $\mathcal{C}_{\mathsf{fin}}(\mathcal{P})$ of $\mathbb{Z}_2$-graded complexes in the category of projectives $\mathcal{P} \subset \mathcal{R}$ with finite homology. In contrast to Bridgeland's construction, however, it is not clear that the corresponding product in the Hall algebra $\mathcal{H}(\mathcal{C}_{\mathsf{fin}}(\mathcal{P}))$ is associative. Therefore the proof of associativity for the (localized) Hall algebra is one of the main results of this paper.

Following the approach of [4], we work in the more general setting of a category $\mathcal{R}$ satisfying certain natural assumptions listed in the next subsection (Section 1.1) which we keep throughout Sections 2-4. The main theorem (Theorem 4.6) for this general setting states that a certain localization $\mathcal{DH}(\mathcal{R})$ of the Hall algebra $\mathcal{H}(\mathcal{C}_{\mathsf{fin}}(\mathcal{P}))$ is isomorphic to the Drinfeld double of the (extended) Hall algebra of $\mathcal{R}$, generalizing a result of Yanagida [23]. As a corollary (Corollary 4.7), the localized Hall algebra is shown to be an associative algebra.

In Section 5, we show that the category $\mathcal{R}$ of finitely-presented representations of a locally finite quiver satisfies all the assumptions in Section 1.1 under some minor restrictions on the quiver. As a consequence of the results of Section 4 in the general setting, we obtain the main result for the quantum group (Theorem 5.12) which establishes an embedding

$$\Xi\colon \mathbf{U}_v \hookrightarrow \mathcal{DH}_{\mathsf{red}}(\mathcal{R})$$

of the whole quantum group $\mathbf{U}_v = \mathbf{U}_v(\mathfrak{g})$ of a generalized Kac–Moody algebra $\mathfrak{g}$ into a reduced version of the localized Hall algebra $\mathcal{DH}(\mathcal{R})$.

1.1. **Assumptions.** Given an abelian category $\mathcal{C}$, let $K(\mathcal{C})$ denote its Grothendieck group and write $\boldsymbol{k}_{\mathcal{C}}(X) \in K(\mathcal{C})$ to denote the class of an object $X \in \mathcal{C}$.

Throughout this paper $\mathcal{R}$ is an abelian category. Let $\mathcal{P} \subset \mathcal{R}$ denote the full subcategory of projectives and $\mathcal{A} \subset \mathcal{R}$ the full subcategory of objects $A \in \mathcal{R}$ such that $\mathrm{Hom}_{\mathcal{R}}(M, A)$ is a finite set for any $M \in \mathcal{R}$. There are several conditions that we will impose on the triple $(\mathcal{R}, \mathcal{P}, \mathcal{A})$. Precisely, we shall always assume that

(a) $\mathcal{R}$ is essentially small and linear over $\Bbbk = \mathbb{F}_q$,
(b) $\mathcal{R}$ is hereditary, that is of global dimension at most 1, and has enough projectives,
(c) for any objects $P, Q, M \in \mathcal{P}$, the relation $M \oplus P \cong M \oplus Q$ implies $P \cong Q$,
(d) every element in $K(\mathcal{R})$ is a $\mathbb{Q}$-linear combination of elements in $\{\boldsymbol{k}_{\mathcal{R}}(A) \mid A \in \mathcal{A}\}$,
(e) the identity $\boldsymbol{k}_{\mathcal{R}}(A) = \boldsymbol{k}_{\mathcal{R}}(B)$ implies $|\mathrm{Hom}(P, A)| = |\mathrm{Hom}(P, B)|$ for all $P \in \mathcal{P}$, $A, B \in \mathcal{A}$.

It is clear that the category $\mathcal{A}$ is Hom-finite, that is $\mathrm{Hom}_{\mathcal{R}}(A,B)$ is a finite set for all $A,B\in\mathcal{A}$. Since $\mathcal{R}$ has enough projectives by (b), it follows that the subcategory $\mathcal{A}$ is abelian and hence a Krull–Schmidt category by [16]. It is also easy to check that $\mathcal{A}$ is closed under extensions in $\mathcal{R}$. We note, however, that the category $\mathcal{R}$ is not necessarily Krull-Schmidt in general.

The condition (c) is required in the proof of Proposition 3.18 to show that the localized Hall algebra $\mathcal{DH}(\mathcal{R})$ is a free module over the group algebra $\mathbb{C}[K(\mathcal{R})\times K(\mathcal{R})]$. Conditions (d) and (e) are needed in Section 2.3 to ensure that the Euler form on $\mathcal{A}$ can be lifted to a bilinear form on the whole category $\mathcal{R}$, albeit with values in $\mathbb{Q}$.

The class of categories $\mathcal{R}$ satisfying the above conditions (a)-(e) generalizes the class of categories $\mathcal{A}$ satisfying Bridgeland's conditions, also denoted (a)-(e) in [4], although our conditions do not correspond precisely. In particular, if the category $\mathcal{R}$ is Hom-finite then $\mathcal{A}=\mathcal{R}$ so that condition (d) is superfluous, and it is also clear that (c) holds since $\mathcal{R}$ is Krull-Schmidt in this case. One may check using Proposition 2.4 that the remaining conditions (a), (b), (e) hold in case $\mathcal{A}=\mathcal{R}$ if and only if the conditions (a)-(e) in [4] hold for $\mathcal{A}$.

1.2. **Notation.** Assume throughout that $\Bbbk=\mathbb{F}_q$ is a finite field ($q\geq 2$) with q elements. Let $v\in\mathbb{R}_{>0}$ be such that $v^2=q$. We denote by $v^{1/n}$ the positive real $2n$-th root of q for $n\in\mathbb{Z}_{\geq 1}$, and write $q^{1/n}=v^{2/n}$. We also write $\mathbb{Z}_2=\mathbb{Z}/2\mathbb{Z}$.

2. Hall algebras

We assume that the triple $(\mathcal{R},\mathcal{P},\mathcal{A})$ satisfies axioms (a)-(e) in Section 1.1.

2.1. **Hall algebras.** Given a small category $\mathcal{C}$, denote by $\mathrm{Iso}(\mathcal{C})$ the set of isomorphism classes in $\mathcal{C}$. Suppose that $\mathcal{C}$ is abelian. Given objects $X,Y,Z\in\mathcal{C}$, define $\mathrm{Ext}^1_{\mathcal{C}}(X,Y)_Z$ to be the set of (equivalence classes of) extensions with middle term isomorphic to Z in $\mathrm{Ext}^1_{\mathcal{C}}(X,Y)$.

Since $\mathcal{R}$ has enough projectives, it follows from the definition of $\mathcal{A}$ that the set $\mathrm{Ext}^1_{\mathcal{A}}(A,B)$ is finite for all $A,B\in\mathcal{A}$. The *Hall algebra* $\mathcal{H}(\mathcal{A})$ is defined to be the $\mathbb{C}$-vector space with basis indexed by elements $A\in\mathrm{Iso}(\mathcal{A})$, and with associative multiplication defined by

$$[A]\diamond[B]=\sum_{C\in\mathrm{Iso}(\mathcal{A})}\frac{|\mathrm{Ext}^1_{\mathcal{A}}(A,B)_C|}{|\mathrm{Hom}_{\mathcal{A}}(A,B)|}\,[C]. \tag{2.1}$$

The unit is given by $[0]$, where 0 is the zero object in $\mathcal{A}$.

Recall from [4] that the multiplication (2.1) is a variant of the usual Hall product (see e.g. [20]) defined as follows. Given objects $A,B,C\in\mathcal{A}$, define the number

$$g^C_{A,B}=\big|\{B'\subset C : B'\cong B,\ C/B'\cong A\}\big|. \tag{2.2}$$

Writing $a_A=|\,\mathrm{Aut}_{\mathcal{A}}(A)|$ to denote the cardinality of the automorphism group of an object A, recall that

$$g^C_{A,B}=\frac{|\mathrm{Ext}^1_{\mathcal{A}}(A,B)_C|}{|\mathrm{Hom}_{\mathcal{A}}(A,B)|}\cdot\frac{a_C}{a_A\,a_B}.$$

Hence using $[[A]]=[A]\cdot a_A^{-1}$ as alternative generators, the product takes the form

$$[[A]]\diamond[[B]]=\sum_{C\in\mathrm{Iso}(\mathcal{A})}g^C_{A,B}\cdot[[C]].$$

The associativity of multiplication in $\mathcal{H}(\mathcal{A})$ then reduces to the equality

$$\sum_{C_1} g_{A,B}^{C_1}\, g_{C_1,C}^{D} \;=\; \sum_{D_1} g_{A,D_1}^{D}\, g_{B,C}^{D_1} \tag{2.3}$$

which holds for any $A, B, C, D \in \mathrm{Iso}(\mathcal{A})$.

2.2. **Twisted and Extended Hall algebras.** The Euler form on $\mathcal{A}$ is a bilinear mapping

$$\langle -, - \rangle : \mathcal{A} \times \mathcal{A} \to \mathbb{Z}$$

defined by

$$\langle A, B \rangle := \dim_{\Bbbk} \mathrm{Hom}(A, B) - \dim_{\Bbbk} \mathrm{Ext}^1(A, B) \tag{2.4}$$

for all $A, B \in \mathcal{A}$. This form factors through the Grothendieck group $K(\mathcal{A})$ (see Lemma 2.1). We also introduce the associated symmetric form $(A, B) = \langle A, B \rangle + \langle B, A \rangle$.

We define the *twisted Hall algebra* $\mathcal{H}_v(\mathcal{A})$ to be the same as the Hall algebra of $\mathcal{A}$ with a new multiplication given by

$$[A] * [B] = v^{\langle \boldsymbol{k}_{\mathcal{A}}(A),\, \boldsymbol{k}_{\mathcal{A}}(B) \rangle} [A] \diamond [B]. \tag{2.5}$$

We extend $\mathcal{H}_v(\mathcal{A})$ by adjoining new generators K_α for all $\alpha \in K(\mathcal{A})$ with the relations

$$K_\alpha * K_\beta = K_{\alpha+\beta}, \qquad K_\alpha * [A] * K_{-\alpha} = \; v^{(\alpha,\, \boldsymbol{k}_{\mathcal{A}}(A))}\, [A], \tag{2.6}$$

where we use the symmetric form $(\, , \,)$. The resulting algebra will be called the *extended Hall algebra* and denoted by $\widetilde{\mathcal{H}_v}(\mathcal{A})$.

2.3. **Generalized Euler form.** Since the category $\mathcal{R}$ has enough projectives, it follows from the definition of $\mathcal{A}$ that $\mathrm{Ext}^1_{\mathcal{R}}(M, A)$ is a finite set for any $M \in \mathcal{R}$ and $A \in \mathcal{A}$. Define an Euler form $\langle -, - \rangle : \mathcal{R} \times \mathcal{A} \to \mathbb{Z}$ by setting

$$\langle M, A \rangle := \dim_{\Bbbk} \mathrm{Hom}_{\mathcal{R}}(M, A) - \dim_{\Bbbk} \mathrm{Ext}^1_{\mathcal{R}}(M, A)$$

for all $M \in \mathcal{R}, A \in \mathcal{A}$. The following lemma shows that the Euler form induces a bilinear map

$$\langle -, - \rangle : K(\mathcal{R}) \times K(\mathcal{A}) \to \mathbb{Z}.$$

Lemma 2.1. *The Euler form $\langle M, A \rangle$ depends only on the classes of objects $M \in \mathcal{R}$ and $A \in \mathcal{A}$ in the Grothendieck groups $K(\mathcal{R})$ and $K(\mathcal{A})$, respectively.*

Proof. Suppose $0 \to M' \to M \to M'' \to 0$ is an exact sequence in $\mathcal{R}$. Then there is a long exact sequence

$$0 \to \mathrm{Hom}(M'', A) \to \mathrm{Hom}(M, A) \to \mathrm{Hom}(M', A)$$

$$\to \mathrm{Ext}(M'', A) \to \mathrm{Ext}(M, A) \to \mathrm{Ext}(M', A) \to 0$$

which shows that

$$\langle M'', A \rangle - \langle M, A \rangle + \langle M', A \rangle = 0.$$

So the Euler form is well-defined on the class of M. The proof for the class of A is similar. $\square$

In the remainder, let us write $\hat{X} = \boldsymbol{k}_{\mathcal{R}}(X)$ to denote the class of an object X in $K(\mathcal{R})$, and continue to write $\boldsymbol{k}_{\mathcal{A}}(-)$ for classes in $K(\mathcal{A})$. The inclusion $\mathcal{A} \subset \mathcal{R}$ induces a canonical map

$$K(\mathcal{A}) \to K(\mathcal{R}), \quad \boldsymbol{k}_{\mathcal{A}}(A) \mapsto \hat{A}. \tag{2.7}$$

Write $\bar{K}(\mathcal{A}) \subset K(\mathcal{R})$ to denote the image of $K(\mathcal{A})$ under this map.

Suppose that $\{P_i\}_{i\in I}$ is a complete list of isomorphism classes of indecomposable projective objects in $\mathcal{P}$, for some indexing set I. Then define the *dimension vector* of an object $A \in \mathcal{A}$ to be

$$\mathbf{dim}\, A := (\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{R}}(P_i, A))_{i\in I} \in \mathbb{Z}^{\oplus I}.$$

From condition (e), we have

$$\mathbf{dim}\, A = \mathbf{dim}\, B$$

for any objects $A, B \in \mathcal{A}$, whenever the classes $\hat{A} = \hat{B}$ are equal in the Grothendieck group $K(\mathcal{R})$.

Lemma 2.2. *Suppose that $A, A' \in \mathcal{A}$. Then $\langle M, A\rangle = \langle M, A'\rangle$ for all $M \in \mathcal{R}$ if and only if*

$$\dim_{\Bbbk} \mathrm{Hom}(P, A) = \dim_{\Bbbk} \mathrm{Hom}(P, A')$$

for all $P \in \mathcal{P}$.

Proof. The only-if-part follows from the fact that $\langle P, A\rangle = \dim_{\Bbbk} \mathrm{Hom}(P, A)$ for all $P \in \mathcal{P}$ and $A \in \mathcal{A}$. For the if-part, let $M \in \mathcal{R}$, $A \in \mathcal{A}$, and suppose $0 \to P \to Q \to M \to 0$ is a projective resolution. Then there is a long exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{R}}(M, A) \longrightarrow \mathrm{Hom}_{\mathcal{R}}(Q, A) \longrightarrow \mathrm{Hom}_{\mathcal{R}}(P, A) \longrightarrow \mathrm{Ext}_{\mathcal{R}}(M, A) \longrightarrow 0,$$

which shows that

$$\langle M, A\rangle = \dim_{\Bbbk} \mathrm{Hom}(Q, A) - \dim_{\Bbbk} \mathrm{Hom}(P, A).$$

The converse statement now follows. □

The above lemma now has the following consequence.

Corollary 2.3. *The Euler form on $\mathcal{R} \times \mathcal{A}$ factors through a bilinear form*

$$\langle -, -\rangle : K(\mathcal{R}) \times \bar{K}(\mathcal{A}) \to \mathbb{Z}.$$

Now suppose $X, Y \in \mathcal{R}$. Then it follows from condition (d) that there exist objects $A_1, \ldots, A_r \in \mathcal{A}$ such that

$$\hat{Y} = c_1\hat{A}_1 + \cdots + c_r\hat{A}_r$$

for some coefficients $c_i \in \mathbb{Q}$. Define a bilinear form

$$K(\mathcal{R}) \times K(\mathcal{R}) \to \mathbb{Q}$$

by setting

$$\langle \hat{X}, \hat{Y}\rangle = \sum_i c_i \cdot \langle \hat{X}, \hat{A}_i\rangle. \tag{2.8}$$

and extend it through linearity. We again define a symmetric version by setting $(\hat{X}, \hat{Y}) = \langle \hat{X}, \hat{Y}\rangle + \langle \hat{Y}, \hat{X}\rangle$ for any $X, Y \in \mathcal{R}$.

Proposition 2.4. *Let A be a nonzero object in $\mathcal{A}$. Then the classes $\boldsymbol{k}_{\mathcal{A}}(A) \in K(\mathcal{A})$ and $\hat{A} \in K(\mathcal{R})$ are both nonzero.*

Proof. Consider a nonzero object $A \in \mathcal{A}$. Since $\mathcal{R}$ has enough projectives, there exists a surjection $P \to A \to 0$ for some projective $P \in \mathcal{P}$. It follows that $\mathrm{Hom}(P, A)$ is a nonzero set. We obtain

$$\langle P, A\rangle = \dim_{\Bbbk} \mathrm{Hom}(P, A) - \dim_{\Bbbk} \mathrm{Ext}(P, A) = \dim_{\Bbbk} \mathrm{Hom}(P, A) \geq 1.$$

By Lemma 2.2 and Corollary 2.3, we obtain

$$\langle P, A\rangle = \langle \hat{P}, \boldsymbol{k}_{\mathcal{A}}(A)\rangle = \langle \hat{P}, \hat{A}\rangle \geq 1.$$

Thus neither $\hat{A}$ nor $\boldsymbol{k}_{\mathcal{A}}(A)$ is zero. □

Denote by $K_{\geq 0}(\mathcal{A}) \subset K(\mathcal{A})$ the positive cone in the Grothendieck group generated by the classes $\boldsymbol{k}_{\mathcal{A}}(A)$ for $A \in \mathcal{A}$. Define

$$\alpha \leq \beta \iff \beta - \alpha \in K_{\geq 0}(\mathcal{A}). \tag{2.9}$$

Then it follows from Proposition 2.4 that $\leq$ is a partial order on $K(\mathcal{A})$.

2.4. **The extended Hall algebra $\widetilde{\mathcal{H}_v}(\mathcal{R})$.** The algebra $\mathcal{H}_v(\mathcal{A})$ is naturally graded by the Grothendieck group $K(\mathcal{R})$:

$$\mathcal{H}_v(\mathcal{A}) = \bigoplus_{\alpha \in K(\mathcal{R})} \mathcal{H}_v(\mathcal{A})_{(\alpha)}, \qquad \mathcal{H}_v(\mathcal{A})_{(\alpha)} := \bigoplus_{\hat{A}=\alpha} \mathbb{C}[A].$$

We define a slight modification of the extended Hall algebra $\widetilde{\mathcal{H}_v}(\mathcal{A})$ from Section 2.2. Starting again from $\mathcal{H}_v(\mathcal{A})$ with multiplication (2.5), the *extended Hall algebra* $\widetilde{\mathcal{H}_v}(\mathcal{R})$ is defined by adjoining generators K_α for all $\alpha \in K(\mathcal{R})$ with the relations

$$K_\alpha * K_\beta = K_{\alpha+\beta}, \qquad K_\alpha * [A] * K_{-\alpha} = v^{(\alpha, \hat{A})}[A]. \tag{2.10}$$

The algebra $\widetilde{\mathcal{H}_v}(\mathcal{R})$ is also $K(\mathcal{R})$-graded, with the degree of each K_α equal to zero. Note that the multiplication map

$$\mathcal{H}_v(\mathcal{A}) \otimes_{\mathbb{C}} \mathbb{C}[K(\mathcal{R})] \to \widetilde{\mathcal{H}_v}(\mathcal{R})$$

is an isomorphism of vector spaces.

Following Green [9] and Xiao [22], we define a coalgebra structure on $\widetilde{\mathcal{H}_v}(\mathcal{R})$. We refer to [23] for the definition of a *topological* coalgebra, which involves a completed tensor product.

Definition 2.5. [9, 22] In the extended Hall algebra $\mathcal{H} = \widetilde{\mathcal{H}_v}(\mathcal{R})$ define $\Delta : \mathcal{H} \to \mathcal{H} \widehat{\otimes}_{\mathbb{C}} \mathcal{H}$, and $\epsilon : \mathcal{H} \to \mathbb{C}$, by setting

$$\Delta([A]K_\alpha) := \sum_{B,C \in \mathrm{Iso}(\mathcal{A})} v^{\langle B,C\rangle} g^A_{B,C} \cdot ([B]K_{\hat{C}+\alpha}) \otimes ([C]K_\alpha), \qquad \epsilon([A]K_\alpha) := \delta_{A,0}$$

for all $A \in \mathrm{Iso}(\mathcal{A})$, $\alpha \in K(\mathcal{R})$, where the numbers $g^A_{B,C}$ are defined in (2.2) and $\mathcal{H} \widehat{\otimes}_{\mathbb{C}} \mathcal{H}$ is the completed tensor product. This gives $\widetilde{\mathcal{H}_v}(\mathcal{R})$ the structure of a topological coassociative coalgebra.

As noted in [21, Remark 1.6], the coproduct Δ on $\mathcal{H} = \widetilde{\mathcal{H}_v}(\mathcal{R})$ takes values in $\mathcal{H} \otimes \mathcal{H}$, instead of the completion $\mathcal{H} \widehat{\otimes} \mathcal{H}$, if and only if the following condition holds:

(2.11) *Any fixed object $A \in \mathcal{A} \subset \mathcal{R}$ has only finitely many subobjects $B \subset A$.*

This condition is satisfied if $\mathcal{R}$ is the category of quiver representations considered in Section 5.

Now we have an algebra structure and a coalgebra structure on $\widetilde{\mathcal{H}_v}(\mathcal{R})$. It follows from [9, 22] that these structures are compatible to give a (topological) *bialgebra* structure. Below, we simply denote by $\widetilde{\mathcal{H}_v}(\mathcal{R})$ the bialgebra $(\widetilde{\mathcal{H}_v}(\mathcal{R}), *, [0], \Delta, \epsilon)$. The bialgebra $\widetilde{\mathcal{H}_v}(\mathcal{R})$ admits a natural bilinear form compatible with the bialgebra structure called a *Hopf pairing*.

Definition 2.6 ([9, 22, 23])**.** Define a bilinear form $(\cdot,\cdot)_H$ on $\widetilde{\mathcal{H}_v}(\mathcal{R})$ by setting

$$([A]K_\alpha, [B]K_\beta)_H := v^{(\alpha,\beta)} a_A\, \delta_{A,B}$$

for $\alpha, \beta \in K(\mathcal{R})$, where $a_A = |\operatorname{Aut}_{\mathcal{A}}(A)|$ as before.

It is clear that the restriction of this bilinear form to the subalgebra $\mathcal{H}_v(\mathcal{A}) \subset \widetilde{\mathcal{H}_v}(\mathcal{R})$ is nondegenerate. The following result was stated for $\widetilde{\mathcal{H}_v}(\mathcal{A})$ in [22]. It is easy to check that it holds for $\widetilde{\mathcal{H}_v}(\mathcal{R})$ as well.

Proposition 2.7 ([21, 22])**.** *The bilinear form $(\cdot,\cdot)_H$ is a Hopf pairing on the bialgebra $\widetilde{\mathcal{H}_v}(\mathcal{R})$, that is, for any $x, y, z \in \widetilde{\mathcal{H}_v}(\mathcal{R})$, one has*

$$(1, x)_H = \epsilon(x), \qquad (x * y, z)_H = (x \otimes y, \Delta(z))_H$$

where we use the usual pairing on the tensor product space:

$$(x \otimes y, z \otimes w)_H = (x, z)_H\, (y, w)_H.$$

2.5. **The Drinfeld double.** We briefly recall the Drinfeld double construction for Hall bialgebras. A complete treatment of Drinfeld doubles is given in [10, §3.2] and [21, §5.2].

In [22], Xiao showed that the extended Hall algebra $\widetilde{H_v}(\mathcal{A})$ is a Hopf algebra and gave an explicit formula for both the antipode σ and its inverse σ^{-1}, provided that $\mathcal{A}$ is a category of quiver representations. It can be shown that Xiao's formulas hold more generally provided that $\mathcal{A} \subset \mathcal{R}$ satisfies (2.11).

Athough the formula for σ is no longer well-defined in the case where $\mathcal{A}$ does not satisfy (2.11), there is a more general condition that ensures the formula for σ^{-1} is still defined. Recall from [8] that an *anti-equivalence* between two objects $A, B \in \mathcal{A}$ is a pair of strict filtrations

$$0 = L_{n+1} \subsetneq L_n \subsetneq \dots L_1 \subsetneq L_0 = A, \qquad 0 = L'_{n+1} \subsetneq L'_n \subsetneq \dots L'_1 \subsetneq L'_0 = B$$

such that $L'_i/L'_{i+1} \cong L_{n-i}/L_{n-i+1}$ for all i. Two objects A and B in $\mathcal{A}$ are called *anti-equivalent* if there exists at least one anti-equivalence between them.

It follows from results in [5] and [8] that Xiao's formula for the map σ^{-1} is still well-defined provided that the following condition holds for every pair of objects $A, B \in \mathcal{A}$:

(2.12) *There are finitely many anti-equivalences (if any) between A and B.*

Since the map σ^{-1} generally takes values in a certain completion of $\widetilde{\mathcal{H}_v}(\mathcal{R})$, an extra condition is needed to ensure that corresponding relations in the bialgebra, such as

$$m \circ (\sigma^{-1} \otimes \mathrm{id}) \circ \Delta^{\mathrm{op}} = i \circ \epsilon,$$

are still well-defined. The required condition can be stated as follows:

(2.13) *Given any $A, B \in \mathcal{A}$, there are finitely many pairs (A', B') of anti-equivalent objects such that $A' \hookrightarrow A$ and $B \twoheadrightarrow B'$.*

Up to minor modifications, the formulas for (and corresponding properties of) σ and σ^{-1} continue to hold, respectively, in $\widetilde{H}_v(\mathcal{R})$, whenever they are defined in $\widetilde{H}_v(\mathcal{A})$.

If the conditions (2.12) and (2.13) are both satisfied by the category $\mathcal{A} \subset \mathcal{R}$, then the *Drinfeld double* of $\mathcal{H} = \widetilde{\mathcal{H}_v}(\mathcal{R})$ is the vector space $\mathcal{H} \otimes \mathcal{H}$ equipped with the multiplication $\circ$ uniquely determined by the following conditions:

(D1) The maps

$$\mathcal{H} \longrightarrow \mathcal{H} \otimes_{\mathbb{C}} \mathcal{H}, \quad a \longmapsto a \otimes 1$$

and

$$\mathcal{H} \longrightarrow \mathcal{H} \otimes_{\mathbb{C}} \mathcal{H}, \quad a \longmapsto 1 \otimes a$$

are injective homomorphisms of $\mathbb{C}$-algebras;

(D2) For all elements $a, b \in \mathcal{H}$, one has

$$(a \otimes 1) \circ (1 \otimes b) = a \otimes b;$$

(D3) For all elements $a, b \in \mathcal{H}$, one has

$$(1 \otimes b) \circ (a \otimes 1) = \sum (b_{(1)}, a_{(3)})_H (\sigma^{-1}(b_{(3)}), a_{(1)})_H \cdot a_{(2)} \otimes b_{(2)}$$

where $\Delta^2(a) = \sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$ and $\Delta^2(b) = \sum b_{(1)} \otimes b_{(2)} \otimes b_{(3)}$.

The last identity is equivalent to

$$\sum (a_{(2)}, b_{(1)})_H \cdot a_{(1)} \otimes b_{(2)} = \sum (a_{(1)}, b_{(2)})_H \cdot (1 \otimes b_{(1)}) \circ (a_{(2)} \otimes 1) \tag{2.14}$$

for all $a, b \in \mathcal{H}$, where $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$ and $\Delta(b) = \sum b_{(1)} \otimes b_{(2)}$.

An argument similar to the proof given in [10, Lemma 3.2.2] shows that the multiplication $\circ$ is associative.

3. Hall algebras of complexes

Assume that $\mathcal{R}$ is an abelian category for which the triple $(\mathcal{R}, \mathcal{P}, \mathcal{A})$ satisfies axioms (a)-(e) of Section 1.1. We now introduce certain categories of complexes over $\mathcal{R}$ and define corresponding Hall algebras and their localizations. The associativity of multiplication in the localized Hall algebras will be established later in Section 4.

3.1. **Categories of complexes.** Define a $\mathbb{Z}_2$-graded chain complex in $\mathcal{R}$ to be a diagram

$$M_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} M_0$$

such that $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}_2$.

A morphism $s_\bullet \colon M_\bullet \to \tilde{M}_\bullet$ consists of a diagram

$$\begin{tikzcd} M_1 \arrow[r, shift left, "d_1"] \arrow[d, "s_1"'] & M_0 \arrow[l, shift left, "d_0"] \arrow[d, "s_0"] \\ \tilde{M}_1 \arrow[r, shift left, "\tilde{d}_1"] & \tilde{M}_0 \arrow[l, shift left, "\tilde{d}_0"] \end{tikzcd}$$

with $s_{i+1} \circ d_i = \tilde{d}_i \circ s_i$.

Let $\mathcal{C}_{\mathbb{Z}_2}(\mathcal{R})$ denote the category of all $\mathbb{Z}_2$-graded chain complexes in $\mathcal{R}$ with morphisms defined above. Two morphisms $s_\bullet, t_\bullet \colon M_\bullet \to \tilde{M}_\bullet$ are *homotopic* if there are morphisms $h_i \colon M_i \to \tilde{M}_{i+1}$ such that

$$t_i - s_i = \tilde{d}_{i+1} \circ h_i + h_{i+1} \circ d_i.$$

We write $\mathrm{Ho}_{\mathbb{Z}_2}(\mathcal{R})$ for the category obtained from $\mathcal{C}_{\mathbb{Z}_2}(\mathcal{R})$ by identifying homotopic morphisms.

The *shift functor* defines an involution

$$\mathcal{C}_{\mathbb{Z}_2}(\mathcal{R}) \qquad \overset{*}{\longleftrightarrow} \qquad \mathcal{C}_{\mathbb{Z}_2}(\mathcal{R})$$

which shifts the grading and changes the sign of the differential

$$M_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} M_0 \qquad \overset{*}{\longleftrightarrow} \qquad M_0 \underset{-d_1}{\overset{-d_0}{\rightleftarrows}} M_1.$$

The image of $M_\bullet$ under the shift functor will be denoted by $M_\bullet^*$. Every complex $M_\bullet \in \mathcal{C}_{\mathbb{Z}_2}(\mathcal{R})$ defines a class $\hat{M}_\bullet := \hat{M}_0 - \hat{M}_1 \in K(\mathcal{R})$ in the Grothendieck group of $\mathcal{R}$.

We are mostly concerned with the full subcategories

$$\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P}),\ \mathcal{C}_{\mathbb{Z}_2}(\mathcal{A}) \subset \mathcal{C}_{\mathbb{Z}_2}(\mathcal{R}) \quad \text{and} \quad \mathrm{Ho}_{\mathbb{Z}_2}(\mathcal{P}) \subset \mathrm{Ho}_{\mathbb{Z}_2}(\mathcal{R})$$

consisting of complexes of objects in $\mathcal{P}$ and $\mathcal{A}$, respectively.

3.2. **Root category.** Let $\mathrm{D}^b(\mathcal{R})$ denote the ($\mathbb{Z}$-graded) bounded derived category of $\mathcal{R}$, with its shift functor [1]. Let $\mathrm{Rt}(\mathcal{R}) = \mathrm{D}^b(\mathcal{R})/[2]$ be the *orbit category*, also known as the *root category* of $\mathcal{R}$. This has the same objects as $\mathrm{D}^b(\mathcal{R})$, but the morphisms are given by

$$\mathrm{Hom}_{\mathrm{Rt}(\mathcal{R})}(X, Y) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{D}^b(\mathcal{R})}(X, Y[2i]).$$

Since $\mathcal{R}$ is an abelian category of finite global dimension (≤ 1) with enough projectives, the category $\mathrm{D}^b(\mathcal{R})$ is equivalent to the ($\mathbb{Z}$-graded) bounded homotopy category $\mathrm{Ho}^b(\mathcal{P})$. Thus we can equally well define $\mathrm{Rt}(\mathcal{R})$ as the orbit category of $\mathrm{Ho}^b(\mathcal{P})$.

Lemma 3.1 ([4])**.** *There is a fully faithful functor*

$$D\colon\ \mathrm{Rt}(\mathcal{R}) \longrightarrow \mathrm{Ho}_{\mathbb{Z}_2}(\mathcal{P})$$

sending a $\mathbb{Z}$-graded complex of projectives $(P_i)_{i\in\mathbb{Z}}$ to the $\mathbb{Z}_2$-graded complex

$$\textstyle\bigoplus_{i\in\mathbb{Z}} P_{2i+1} \underset{0}{\overset{0}{\rightleftarrows}} \bigoplus_{i\in\mathbb{Z}} P_{2i}.$$

3.3. **Decompositions.** From now on, we omit $\mathbb{Z}_2$ in the notations for categories and write

$$\mathcal{C}(\mathcal{P}) = \mathcal{C}_{\mathbb{Z}_2}(\mathcal{P}), \qquad \mathcal{C}(\mathcal{A}) = \mathcal{C}_{\mathbb{Z}_2}(\mathcal{A}), \qquad \mathcal{C}(\mathcal{R}) = \mathcal{C}_{\mathbb{Z}_2}(\mathcal{R}),$$
$$\mathrm{Ho}(\mathcal{P}) = \mathrm{Ho}_{\mathbb{Z}_2}(\mathcal{P}), \qquad \mathrm{Ho}(\mathcal{R}) = \mathrm{Ho}_{\mathbb{Z}_2}(\mathcal{R}).$$

The homology of a complex $M_\bullet \in \mathcal{C}(\mathcal{R})$ will be denoted

$$H_\bullet(M_\bullet) \ :=\ (H_1(M_\bullet) \underset{0}{\overset{0}{\rightleftarrows}} H_0(M_\bullet)) \ \in \mathcal{C}(\mathcal{R}).$$

To each morphism $f : P \to Q$ in the category $\mathcal{P}$, we associate the following complexes

$$C_f := (P \underset{0}{\overset{f}{\rightleftarrows}} Q), \qquad C_f^* := (Q \underset{-f}{\overset{0}{\rightleftarrows}} P) \tag{3.1}$$

in $\mathcal{C}(\mathcal{P})$.

Lemma 3.2. *Every complex of projectives $M_\bullet \in \mathcal{C}(\mathcal{P})$ can be decomposed uniquely, up to isomorphism, as a direct sum of complexes of the form*

$$M_\bullet = C_f \oplus C_g^*$$

for some injective morphisms f, g in $\mathcal{P}$ such that $H_0(M_\bullet) = \operatorname{coker}(f)$ and $H_1(M_\bullet) = \operatorname{coker}(g)$.

Proof. Consider the short exact sequences

$$0 \longrightarrow \ker(d_1) \xrightarrow{i} M_1 \xrightarrow{p} \operatorname{im}(d_1) \longrightarrow 0,$$

$$0 \longrightarrow \ker(d_0) \xrightarrow{j} M_0 \xrightarrow{q} \operatorname{im}(d_0) \longrightarrow 0.$$

Since the the category $\mathcal{R}$ is hereditary by assumption, all the objects appearing in these sequences are projective. Thus the sequences split, and we can find morphisms

$$r\colon M_1 \to \ker(d_1), \quad k\colon \operatorname{im}(d_0) \to M_0, \quad l\colon \operatorname{im}(d_1) \to M_1, \quad s\colon M_0 \to \ker(d_0)$$

such that $r \circ i = \mathrm{id}$, $q \circ k = \mathrm{id}$, $p \circ l = \mathrm{id}$, and $s \circ j = \mathrm{id}$. This yields the following split exact sequence of morphisms of complexes

$$\begin{tikzcd}
\ker(d_1) \arrow[r, shift left, "0"] \arrow[d, "i"'] & \operatorname{im}(d_0) \arrow[l, shift left, "m"] \arrow[d, "k"] \\
M_1 \arrow[r, shift left, "d_1"] \arrow[d, "p"'] & M_0 \arrow[l, shift left, "d_0"] \arrow[d, "s"] \\
\operatorname{im}(d_1) \arrow[r, shift left, "m'"] & \ker(d_0) \arrow[l, shift left, "0"]
\end{tikzcd}$$

where m, m' denote the obvious inclusions. (Note that $d_0 = i \circ m \circ q$ and $d_1 = j \circ m' \circ p$.) The desired decomposition of $M_\bullet$ is thus given by setting: $f = m \circ q$, $g = -m' \circ l$.

Now suppose there is an isomorphism $M_\bullet \cong C_{f'} \oplus C^*_{g'}$ for some other pair f', g' of injective morphisms in $\mathcal{P}$. Then one can easily define corresponding isomorphisms of complexes $C_{f'} \cong C_f$ and $C_{g'} \cong C_g$, showing uniqueness. □

Given $M_\bullet \in \mathcal{C}(\mathcal{P})$, it will be convenient to write the decomposition in Lemma 3.2 as

$$M_\bullet = M_\bullet^+ \oplus M_\bullet^-$$

where $M_\bullet^+ = C_f$ and $M_\bullet^- = C_g^*$. Let the sign map

$$\varepsilon : \mathbb{Z}_2 \to \{+, -\}$$

be defined by $\varepsilon(0) = +$, $\varepsilon(1) = -$.

Lemma 3.3. *Let $M_\bullet, N_\bullet \in \mathcal{C}(\mathcal{P})$. Then there is an isomorphism*

$$\operatorname{Hom}_{\mathcal{C}(\mathcal{R})}(M_\bullet, N_\bullet) \;\cong\; \operatorname{Hom}_{\mathcal{C}(\mathcal{R})}(H_\bullet(M_\bullet), H_\bullet(N_\bullet)) \oplus \Big\{ \bigoplus_{i,j \in \mathbb{Z}_2} \operatorname{Hom}_{\mathcal{R}}(M_i^{\varepsilon(j)}, N_{j+1}^{\varepsilon(i)}) \Big\}$$

of $\Bbbk$-vector spaces.

Proof. First suppose that $P \xrightarrow{f} Q$, $P' \xrightarrow{g} Q'$ is a pair of injective morphisms in $\mathcal{P}$. We note that there is a short exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{R}}(Q, P') \longrightarrow \mathrm{Hom}_{\mathcal{C}(\mathcal{R})}(C_f, C_g) \longrightarrow \mathrm{Hom}_{\mathcal{R}}(X, Y) \longrightarrow 0 \tag{3.2}$$

for $X = \operatorname{coker} f$, $Y = \operatorname{coker} g$. One may also check directly that

$$\mathrm{Hom}_{\mathcal{C}(\mathcal{R})}(C_f, C_g^*) \cong \mathrm{Hom}_{\mathcal{R}}(P, Q'). \tag{3.3}$$

The decomposition of $\mathrm{Hom}_{\mathcal{C}(\mathcal{R})}(M_\bullet, N_\bullet)$ now follows easily by applying Lemma 3.2 to both $M_\bullet$ and $N_\bullet$, respectively, and by using the involutive shift functor $*$ together with (3.2) and (3.3). □

3.4. **Acyclic complexes.** Given a projective object $P \in \mathcal{P}$, there are associated acyclic complexes

$$K_P := (P \underset{0}{\overset{\mathrm{id}}{\rightleftarrows}} P), \qquad K_P^* := (P \underset{-\mathrm{id}}{\overset{0}{\rightleftarrows}} P). \tag{3.4}$$

Notice that $M_\bullet \in \mathcal{C}(\mathcal{P})$ is acyclic precisely if $M_\bullet \cong 0$ in $\mathrm{Ho}(\mathcal{P})$.

Lemma 3.4. *If $M_\bullet \in \mathcal{C}(\mathcal{P})$ is an acyclic complexes of projectives, then there exist objects $P, Q \in \mathcal{P}$, unique up to isomorphism, such that $M_\bullet \cong K_P \oplus K_Q^*$.*

Proof. If $M_\bullet$ is acyclic, then by Lemma 3.2 we have $M_\bullet = C_f \oplus C_g^*$ for some isomorphisms $f : P \cong P'$ and $g : Q \cong Q'$ of projectives. It follows that $M_\bullet \cong K_P \oplus K_Q^*$. Since the complexes K_P and K_Q^* are unique up to isomorphism, the objects P, Q are unique up to isomorphism as well. □

Lemma 3.5. *Suppose that $P \xrightarrow{f} Q$ and $P' \xrightarrow{f'} Q'$ are injective morphisms in $\mathcal{P}$. Then* $\operatorname{coker} f \cong \operatorname{coker} f'$ *in $\mathcal{R}$, if and only if there is an isomorphism*

$$C_f \oplus K_{L'} \;\cong\; K_L \oplus C_{f'}$$

of complexes in $\mathcal{C}(\mathcal{P})$, for some objects $L, L' \in \mathcal{P}$.

Proof. This is a reformulation of Schanuel's lemma. We refer to [7, Theorem 0.5.3] for the proof. □

Proposition 3.6. *Suppose $M_\bullet, M'_\bullet \in \mathcal{C}(\mathcal{P})$. Then there exists an isomorphism*

$$M_\bullet \oplus K'_\bullet \;\cong\; K_\bullet \oplus M'_\bullet$$

for some acyclic complexes $K_\bullet, K'_\bullet \in \mathcal{C}(\mathcal{P})$, if and only if $H_\bullet(M_\bullet) \cong H_\bullet(M'_\bullet)$ in $\mathcal{C}(\mathcal{R})$.

Proof. This follows directly from Lemmas 3.2 and 3.5, and by applying $*$ to the latter. □

3.5. **Extensions of complexes.** Given any morphism $s_\bullet : M_\bullet \to N_\bullet$ of complexes in $\mathcal{C}(\mathcal{P})$, we can form a corresponding exact sequence

$$0 \longrightarrow N_\bullet^* \longrightarrow \mathrm{Cone}(s_\bullet) \longrightarrow M_\bullet \longrightarrow 0$$

of complexes in $\mathcal{C}(\mathcal{P})$, where the middle term is defined by

$$\mathrm{Cone}(s_\bullet) = (N_0 \oplus M_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} N_1 \oplus M_0)$$

with

$$d_0 := \begin{bmatrix} -d_1^N & s_0 \\ 0 & d_0^M \end{bmatrix}, \qquad d_1 := \begin{bmatrix} -d_0^N & s_1 \\ 0 & d_1^M \end{bmatrix}.$$

This leads to the following result.

Lemma 3.7 ([4]). *Let $M_\bullet, N_\bullet \in \mathcal{C}(\mathcal{P})$. The mapping $s_\bullet \mapsto \mathrm{Cone}(s_\bullet)$ defines an isomorphism*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{R})}(M_\bullet, N_\bullet^*) \;\cong\; \mathrm{Ext}^1_{\mathcal{C}(\mathcal{R})}(M_\bullet, N_\bullet).$$

We also have the following.

Lemma 3.8. *Suppose $P \xrightarrow{f} Q$, $P' \xrightarrow{g} Q'$ are injective morphisms in the category $\mathcal{P}$. Let $X, Y \in \mathcal{R}$ denote the cokernels: $X = \operatorname{coker} f$, $Y = \operatorname{coker} g$. Then the following hold.*

(i) $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{R})}(C_f, C_g) \cong \mathrm{Hom}_{\mathcal{R}}(X, Y)$*;*
(ii) $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{R})}(C_f, C_g^*) \cong \mathrm{Ext}^1_{\mathcal{R}}(X, Y)$.

Proof. The category $\mathcal{R}$ can be identified as a full subcategory of $\mathrm{D}^b(\mathcal{R})$ by considering any object in $\mathcal{R}$ as a complex concentrated in degree 0. It follows that

$$\mathrm{Hom}_{\mathcal{R}}(X, Y) \cong \mathrm{Hom}_{\mathrm{D}^b(\mathcal{R})}(X, Y) \cong \mathrm{Hom}_{\mathrm{Rt}(\mathcal{R})}(X, Y).$$

The objects X, Y have the projective resolutions

$$0 \to P \xrightarrow{f} Q \twoheadrightarrow X \to 0, \qquad 0 \to P' \xrightarrow{g} Q' \twoheadrightarrow Y \to 0.$$

So the complexes C_f, C_g are quasi-isomorphic to X, Y respectively, and isomorphisms (i), (ii) thus follow by Lemmas 3.1. □

Note that the isomorphism in Lemma 3.8 (i) may be given explicitly by $s_\bullet \mapsto s$, where $s : A \to B$ is the unique morphism making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P & \longrightarrow & Q & \longrightarrow & X & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle s_1} & & \big\downarrow{\scriptstyle s_0} & & \big\downarrow{\scriptstyle s} & & \\ 0 & \longrightarrow & P' & \longrightarrow & Q' & \longrightarrow & Y & \longrightarrow & 0 \end{array} \tag{3.5}$$

commutative.

3.6. **Hall algebras of complexes.** We denote by $\mathcal{C}_{\mathsf{fin}}(\mathcal{P})$ the full subcategory of $\mathcal{C}(\mathcal{P})$ consisting of all complexes with finite homology, i.e. complexes $M_\bullet$ such that

$$H_\bullet(M_\bullet) \;=\; (H_1(M_\bullet) \underset{0}{\overset{0}{\rightleftarrows}} H_0(M_\bullet)) \;\in \mathcal{C}(\mathcal{A}).$$

The following result will be crucial for our definition of the Hall algebra of $\mathcal{C}_{\mathsf{fin}}(\mathcal{P})$.

Lemma 3.9. *The set $\mathrm{Ext}^1_{\mathcal{C}(\mathcal{R})}(M_\bullet, N_\bullet)$ is finite for all $M_\bullet, N_\bullet \in \mathcal{C}_{\mathsf{fin}}(\mathcal{P})$.*

Proof. This follows by using the involution $*$ together with Lemma 3.8 and combining with Lemmas 3.2 and 3.7. □

Since the category $\mathcal{C}_{\mathsf{fin}}(\mathcal{P})$ is not necessarily Hom-finite, we must consider a generalization of the coefficients appearing in the definition (2.1) of the Hall product. First define a bilinear map, $\mu : \mathcal{C}(\mathcal{P}) \times \mathcal{C}(\mathcal{P}) \to \mathbb{Q}$, given by

$$\begin{aligned}\mu(M_\bullet, N_\bullet) \;&:=\; \langle \hat{M}_0^+, \hat{N}_1^+\rangle + \langle \hat{M}_1^+, \hat{N}_1^-\rangle + \langle \hat{M}_0^-, \hat{N}_0^+\rangle + \langle \hat{M}_1^-, \hat{N}_0^-\rangle \\ &=\; \sum_{i,j} \langle \hat{M}_i^{\varepsilon(j)}, \hat{N}_{j+1}^{\varepsilon(i)}\rangle\end{aligned} \tag{3.6}$$

where $\langle\ ,\ \rangle : K(\mathcal{R}) \times K(\mathcal{R}) \to \mathbb{Q}$ denotes the generalized Euler form (Section 2.3).

Then for any $M_\bullet, N_\bullet, P_\bullet \in \mathcal{C}_{\mathsf{fin}}(\mathcal{P})$, we define

$$\mathrm{h}(M_\bullet, N_\bullet) \;:=\; q^{\mu(M_\bullet, N_\bullet)}\, |\mathrm{Hom}_{\mathcal{C}(\mathcal{R})}(H_\bullet(M_\bullet), H_\bullet(N_\bullet))| \tag{3.7}$$

where q is the cardinality of $\Bbbk$, and we also write

$$\mathrm{e}(M_\bullet, N_\bullet)_{P_\bullet} \;:=\; |\mathrm{Ext}^1_{\mathcal{C}(\mathcal{R})}(M_\bullet, N_\bullet)_{P_\bullet}|$$

which is well-defined by Lemma 3.9.

In the remainder, let us write $\mathcal{X} = \mathrm{Iso}(\mathcal{C}_{\mathsf{fin}}(\mathcal{P}))$ for the set of isomorphism classes in $\mathcal{C}_{\mathsf{fin}}(\mathcal{P})$.

Definition 3.10. The Hall algebra $\mathcal{H}(\mathcal{C}_{\mathsf{fin}}(\mathcal{P}))$ is defined to be the $\mathbb{C}$-vector space with basis elements $[M_\bullet]$ indexed by isoclasses $M_\bullet \in \mathcal{X}$, and with multiplication defined by

$$[M_\bullet] \circledast [N_\bullet] \;:= v^{\langle \hat{M}_0, \hat{N}_0\rangle + \langle \hat{M}_1, \hat{N}_1\rangle} \sum_{P_\bullet \in \mathcal{X}} \frac{\mathrm{e}(M_\bullet, N_\bullet)_{P_\bullet}}{\mathrm{h}(M_\bullet, N_\bullet)}\, [P_\bullet],$$

for all $M_\bullet, N_\bullet \in \mathcal{C}_{\mathsf{fin}}(\mathcal{P})$.

Remark 3.11. Suppose there are complexes $M_\bullet, N_\bullet \in \mathcal{C}_{\mathsf{fin}}(\mathcal{P})$ such that $\mathrm{Hom}_{\mathcal{C}(\mathcal{R})}(M_\bullet, N_\bullet)$ is a finite set. Then it can be checked using Lemma 3.3 and the definition of Euler form that

$$|\mathrm{Hom}_{\mathcal{C}(\mathcal{R})}(M_\bullet, N_\bullet)| = \mathrm{h}(M_\bullet, N_\bullet).$$

The above definition thus generalizes the (twisted) Hall algebras of complexes of projectives defined by Bridgeland in [4].

3.7. **Localization.** As before, let us write $\hat{M}_\bullet = \hat{M}_0 - \hat{M}_1 \in K(\mathcal{P})$, for each $M_\bullet \in \mathcal{C}(\mathcal{P})$. The following result shows that the acyclic complexes K_P introduced in Section 3.4 define elements of $\mathcal{H}(\mathcal{C}_{\mathsf{fin}}(\mathcal{P}))$ with particularly simple properties.

Lemma 3.12. *For any projective object $P \in \mathcal{P}$ and any complex $M_\bullet \in \mathcal{C}_{\mathsf{fin}}(\mathcal{P})$ the following identities hold in $\mathcal{H}(\mathcal{C}_{\mathsf{fin}}(\mathcal{P}))$:*

$$[K_P] \circledast [M_\bullet] = v^{\langle \hat{P}, \hat{M}_\bullet\rangle} \cdot [K_P \oplus M_\bullet], \tag{3.8}$$

$$[M_\bullet] \circledast [K_P] = v^{-\langle \hat{M}_\bullet, \hat{P}\rangle} \cdot [K_P \oplus M_\bullet]. \tag{3.9}$$

Proof. It is easy to check directly from (3.6) that

$$\mu(K_P, M_\bullet) = \langle \hat{P}, \hat{M}_1^+\rangle + \langle \hat{P}, \hat{M}_1^-\rangle, \quad \mu(M_\bullet, K_P) = \langle \hat{M}_0^+, \hat{P}\rangle + \langle \hat{M}_0^-, \hat{P}\rangle$$

so that

$$\mathrm{h}(K_P, M_\bullet) = q^{\langle \hat{P}, \hat{M}_1\rangle}, \quad \mathrm{h}(M_\bullet, K_P) = q^{\langle \hat{M}_0, \hat{P}\rangle}.$$

The complexes K_P are homotopy equivalent to the zero complex, so Lemma 3.7 shows that the extension group in the definition of the Hall product vanishes. Taking into account Definition 3.10 gives the result. $\square$

Lemma 3.13. *For any projective object $P \in \mathcal{P}$ and any complex $M_\bullet \in \mathcal{C}_{\mathsf{fin}}(\mathcal{P})$ the following identities are true in $\mathcal{H}(\mathcal{C}_{\mathsf{fin}}(\mathcal{P}))$:*

$$[K_P] \circledast [M_\bullet] = v^{(\hat{P},\hat{M}_\bullet)}\,[M_\bullet] \circledast [K_P], \tag{3.10}$$

$$[K_P^*] \circledast [M_\bullet] = v^{-(\hat{P},\hat{M}_\bullet)}\,[M_\bullet] \circledast [K_P^*]. \tag{3.11}$$

Proof. Equation (3.10) is immediate from Lemma 3.12. Equation (3.11) follows by applying the involution $*$. □

In particular, since $\hat{K}_P = 0 \in K(\mathcal{R})$, we have for $P, Q \in \mathcal{P}$,

$$[K_P] \circledast [K_Q] = [K_P \oplus K_Q], \quad [K_P] \circledast [K_Q^*] = [K_P \oplus K_Q^*], \tag{3.12}$$

$$\big[[K_P],[K_Q]\big] = \big[[K_P],[K_Q^*]\big] = \big[[K_P^*],[K_Q^*]\big] = 0, \tag{3.13}$$

where $[x, y] := x \circledast y - y \circledast x$. Note that any element of the form $[K_P] \circledast [K_P^*]$ is central.

Let us write $\mathcal{C}_0 \subset \mathcal{C}_{\mathsf{fin}}(\mathcal{P})$ to denote the full subcategory of all acyclic complexes of projectives. It then follows from (3.12) and (3.13) that the subspace $\mathcal{H}(\mathcal{C}_0) \subset \mathcal{H}(\mathcal{C}_{\mathsf{fin}}(\mathcal{P}))$ spanned by the isoclasses of objects in $\mathcal{C}_0$ is closed under the multiplication $\circledast$ and has the structure of a commutative associative algebra.

The following result is also clear.

Lemma 3.14. *The left and right actions of $\mathcal{H}(\mathcal{C}_0)$ on $\mathcal{H}(\mathcal{C}_{\mathsf{fin}}(\mathcal{P}))$ given by restricting multiplication make the Hall algebra $\mathcal{H}(\mathcal{C}_{\mathsf{fin}}(\mathcal{P}))$ into an $\mathcal{H}(\mathcal{C}_0)$-bimodule.*

Notice that the basis $\mathcal{Z} := \{[M_\bullet] \in \mathcal{H}(\mathcal{C}_0)\}$ is a multiplicative subset in $\mathcal{H}(\mathcal{C}_0)$. Let us write $\mathcal{DH}_0(\mathcal{R}) = \mathcal{H}(\mathcal{C}_0)_{\mathcal{Z}}$ to denote the localization of $\mathcal{H}(\mathcal{C}_0)$ at $\mathcal{Z}$. More explicitly, we have

$$\mathcal{DH}_0(\mathcal{R}) = \mathcal{H}(\mathcal{C}_0)\big[[M_\bullet]^{-1} : M_\bullet \in \mathcal{C}_0\big].$$

The assignment $P \mapsto K_P$ extends to a group homomorphism

$$K \colon K(\mathcal{R}) \longrightarrow \mathcal{DH}_0(\mathcal{R})^\times.$$

This map is given explicitly by writing an element $\alpha \in K(\mathcal{R})$ in the form $\alpha = \hat{P} - \hat{Q}$ for objects $P, Q \in \mathcal{P}$ and then setting $K_\alpha := K_{\hat{P}} \circledast K_{\hat{Q}}^{-1}$. Composing with the involution $*$ gives another map

$$K^* \colon K(\mathcal{R}) \longrightarrow \mathcal{DH}_0(\mathcal{R})^\times.$$

Taking these maps together an extending linearly defines a $\mathbb{C}$-linear map from the group algebra

$$\mathbb{C}[K(\mathcal{R}) \times K(\mathcal{R})] \xrightarrow{\sim} \mathcal{DH}_0(\mathcal{R}),$$

which is an isomorphism by Lemma 3.4. It follows that the set $\{K_\alpha \circledast K_\beta^* \mid \alpha, \beta \in K(\mathcal{R})\}$ gives a $\mathbb{C}$-basis of $\mathcal{DH}_0(\mathcal{R})$

Definition 3.15. The *localized Hall algebra*, $\mathcal{DH}(\mathcal{R})$, is the right $\mathcal{DH}_0(\mathcal{R})$-module obtained from $\mathcal{H}(\mathcal{C}_{\mathsf{fin}}(\mathcal{P}))$ by extending scalars,

$$\mathcal{DH}(\mathcal{R}) := \mathcal{H}(\mathcal{C}_{\mathsf{fin}}(\mathcal{P})) \otimes_{\mathcal{H}(\mathcal{C}_0)} \mathcal{DH}_0(\mathcal{R}).$$

That is, $\mathcal{DH}(\mathcal{R})$ is the localization $\mathcal{H}(\mathcal{C}_{\mathsf{fin}}(\mathcal{P}))_{\mathcal{Z}}$. We also consider $\mathcal{DH}(\mathcal{R})$ as a $\mathcal{DH}_0(\mathcal{R})$-bimodule by setting

$$(K_\alpha \circledast K^*_\beta) \circledast [M_\bullet] := v^{(\alpha-\beta,\hat{M}_\bullet)} \cdot [M_\bullet] \circledast (K_\alpha \circledast K^*_\beta) \tag{3.14}$$

for all $M_\bullet \in \mathcal{C}_{\mathsf{fin}}(\mathcal{P})$ and $\alpha, \beta \in K(\mathcal{R})$. We thus have a well-defined binary operation

$$- \circledast - : \mathcal{DH}(\mathcal{R}) \times \mathcal{DH}(\mathcal{R}) \to \mathcal{DH}(\mathcal{R})$$

which agrees with the map induced from the multiplication in Definition 3.10 by restricting along the canonical map $\mathcal{H}(\mathcal{C}_{\mathsf{fin}}(\mathcal{P})) \to \mathcal{DH}(\mathcal{R})$.

Given a complex $M_\bullet \in \mathcal{C}_{\mathsf{fin}}(\mathcal{P})$, define a corresponding element $E_{M_\bullet}$ in $\mathcal{DH}(\mathcal{R})$ given by

$$E_{M_\bullet} := v^{\langle \hat{M}_1^+ - \hat{M}_0^-, \hat{M}_\bullet\rangle}\, K_{-\hat{M}_1^+} \circledast K^*_{-\hat{M}_0^-} \circledast [M_\bullet].$$

Then we claim that

$$E_{M_\bullet \oplus K_\bullet} = E_{M_\bullet} \tag{3.15}$$

for any acyclic complex of projectives $K_\bullet \in \mathcal{C}_0$. Indeed, suppose that $K_\bullet = K_P \oplus K^*_Q$ for some $P, Q \in \mathcal{P}$. Then clearly $M_\bullet \hat{\oplus} K_\bullet = \hat{M}_\bullet$, and it follows by Lemma 3.12 that

$$\begin{aligned} E_{M_\bullet \oplus K_\bullet} &= v^{\langle M_1^+ \hat{\oplus} P - M_0^- \hat{\oplus} Q,\, M_\bullet \hat{\oplus} K_\bullet\rangle} \cdot K_{-M_1^+ \hat{\oplus} P} \circledast K^*_{-M_0^- \hat{\oplus} Q} \circledast [K_P \oplus K^*_Q \oplus M_\bullet] \\ &= v^{\langle \hat{M}_1^+ - \hat{M}_0^-,\, \hat{M}_\bullet\rangle} \cdot K_{-\hat{M}_1^+} \circledast K^*_{-\hat{M}_0^-} \circledast [M_\bullet], \end{aligned}$$

so we get the same element $E_{M_\bullet}$.

We note that a minimal projective resolution of $A \in \mathcal{A}$ need not be unique because the category $\mathcal{C}(\mathcal{P})$ is not Krull–Schmidt in general. However, it will be convenient to fix a (not necessarily minimal) resolution for each object $A \in A$.

Definition 3.16. (i) For each object $A \in \mathcal{A}$, fix a projective resolution

$$0 \longrightarrow P_A \xrightarrow{f_A} Q_A \longrightarrow A \longrightarrow 0 \tag{3.16}$$

and the complex C_A is defined to be $C_{f_A} \in \mathcal{C}(\mathcal{P})$.

(ii) Given objects $A, B \in \mathcal{A}$, write $E_{A,B}$ to denote the element $E_{C_A \oplus C^*_B}$ in $\mathcal{DH}(\mathcal{R})$.

The next lemma shows that the definition of $E_{A,B}$ is independent of the choice of resolutions defining C_A and C_B.

Lemma 3.17. *Suppose $A, B \in \mathcal{A}$, and let $M_\bullet \in \mathcal{C}_{\mathsf{fin}}(\mathcal{P})$ be any complex such that $A \cong H_0(M_\bullet)$ and $B \cong H_1(M_\bullet)$. Then $E_{M_\bullet} = E_{A,B}$.*

Proof. Let $M_\bullet$ be such a complex. By Proposition 3.6 there exist acyclic complexes $K_\bullet, K'_\bullet$ in $\mathcal{C}_0(\mathcal{P})$ such that $[M_\bullet \oplus K_\bullet] \cong [C_A \oplus C^*_B \oplus K'_\bullet]$, and the result follows from (3.15). ☐

The following result provides an explicit basis for the localized Hall algebra.

Proposition 3.18. *The algebra $\mathcal{DH}(\mathcal{R})$ is free as a right $\mathcal{DH}_0(\mathcal{R})$-module, with basis consisting of elements $E_{A,B}$ indexed by all pairs of objects $A, B \in \mathrm{Iso}(\mathcal{A})$.*

Proof. Suppose $M_\bullet \in \mathcal{X}$, and set $A = H_0(M_\bullet)$, $B = H_1(M_\bullet)$. Then $E_{M_\bullet} = E_{A,B}$ by Lemma 3.17, and one may check using (3.14) that

$$[M_\bullet] = v^{\langle \hat{Q}-\hat{P}, \hat{M}_\bullet \rangle}\, E_{A,B} \circledast [K_P \oplus K_Q^*],$$

for $P = M_1^+$ and $Q = M_0^-$. This shows that the elements $E_{A,B}$ span $\mathcal{DH}(\mathcal{R})$ as a $\mathcal{DH}_0(\mathcal{R})$-module.

It remains to check that the elements $E_{A,B}$ are $\mathcal{DH}_0(\mathcal{R})$–linearly independent. Notice that the Hall algebra $\mathcal{H}(\mathcal{C}_{\mathsf{fin}}(\mathcal{P}))$ is naturally graded as a $\mathbb{C}$-vector space by the set $\mathrm{Iso}(\mathcal{A}) \times \mathrm{Iso}(\mathcal{A})$:

$$\mathcal{H}(\mathcal{C}_{\mathsf{fin}}(\mathcal{P})) = \bigoplus_{(A,B)\in \mathrm{Iso}(\mathcal{A})^2} \mathcal{H}_{(A,B)}, \qquad \mathcal{H}_{(A,B)} := \bigoplus_{H_0(M_\bullet)\simeq A,\, H_1(M_\bullet)\simeq B} \mathbb{C}[M_\bullet].$$

Since the action of $\mathcal{H}(\mathcal{C}_0)$ on $\mathcal{H}(\mathcal{C}_{\mathsf{fin}}(\mathcal{P}))$ is $\mathrm{Iso}(\mathcal{A})^2$-homogeneous, it follows that $\mathcal{DH}(\mathcal{R})$ also has an $\mathrm{Iso}(\mathcal{A})^2$-grading. It is thus clear that the elements $E_{A,B}$ span distinct graded components of $\mathcal{DH}(\mathcal{R})$. To see that each component is a free $\mathcal{DH}_0(\mathcal{R})$-module of rank one, it remains to check that for each $y \in \mathcal{DH}(\mathcal{R})$, we have $E_{A,B} \circledast y = 0$ implies $y = 0$.

Let us write $M_\bullet = C_A \oplus C_B^*$. Then it will suffice to show that for any $x \in \mathcal{H}(\mathcal{C}_0)$, the element

$$[M_\bullet] \circledast x \in \mathcal{H}(\mathcal{C}_{\mathsf{fin}}(\mathcal{P}))$$

is a $\mathcal{Z}$-torsion element only if $x = 0$. Suppose that

$$x = c_1 z_1 + \cdots + c_n z_n$$

for some constants $c_1, \ldots, c_n \in \mathbb{C}$ and distinct elements $z_1, \ldots, z_n \in \mathcal{Z}$. One may check using Lemma 3.4 and condition (c) in Section 1.1, that for any $z \in \mathcal{Z}$ the elements $z_1 \circledast z, \ldots, z_n \circledast z$ are also distinct. Next suppose that $[M_\bullet] \circledast (x \circledast z) = 0$ in $\mathcal{H}(\mathcal{C}_{\mathsf{fin}}(\mathcal{P}))$. This gives an equation

$$c_1 \cdot [M_\bullet] \circledast z_1 \circledast z + \cdots + c_n \cdot [M_\bullet] \circledast z_n \circledast z = 0.$$

One may again use condition (c) together with Lemmas 3.2 and 3.4 to check that the terms appearing in this dependence relation are unit multiples of distinct basis elements in $\mathcal{X}$. So the relation must be trivial: $c_1 = \cdots = c_n = 0$, which gives $x = 0$. This completes the proof. □

4. Associativity via the Drinfeld double

In this section we prove that $\mathcal{DH}(\mathcal{R})$ is the Drinfeld double of the bialgebra $\widetilde{\mathcal{H}_v}(\mathcal{R})$ under suitable finiteness conditions. As a corollary, we show that $\mathcal{DH}(\mathcal{R})$ is an associative algebra with respect to the multiplication described in the previous section.

4.1. **Multiplication formulas.** Suppose $A, B \in \mathcal{A}$ and recall the element $E_{A,B}$ in $\mathcal{DH}(\mathcal{R})$ defined in Definition 3.16. Notice that the image under the involution $*$ is given by $E_{A,B}^* = E_{B,A}$. Let us write

$$E_A := E_{A,0}, \qquad F_B := E_{0,B}$$

so that $F_A = E_A^*$.

Lemma 4.1. *Suppose $A, B \in \mathcal{A}$. The following equality holds in $\mathcal{DH}(\mathcal{R})$.*

$$E_A \circledast E_B = v^{\langle A,B \rangle} \sum_{C \in \mathrm{Iso}(\mathcal{R})} \frac{|\mathrm{Ext}^1_{\mathcal{R}}(A,B)_C|}{|\mathrm{Hom}_{\mathcal{R}}(A,B)|}\, E_C$$

Proof. Let C_A, C_B be the complexes associated to A, B in Definition 3.16. Then using the formula

$$\mathrm{h}(C_A, C_B) = q^{\langle \hat{Q}_A, \hat{P}_B \rangle} \cdot |\mathrm{Hom}_{\mathcal{R}}(A, B)|$$

together with the relations $\hat{A} = \hat{Q}_A - \hat{P}_A$ and $\hat{B} = \hat{Q}_B - \hat{P}_B$ in the Grothendieck group $K(\mathcal{R})$, we have

$$[C_A] \circledast [C_B] = v^{\langle \hat{P}_A, \hat{P}_B \rangle + \langle \hat{Q}_A, \hat{Q}_B \rangle} \sum_{M_\bullet \in \mathcal{X}} \frac{\mathrm{e}(C_A, C_B)_{M_\bullet}}{\mathrm{h}(C_A, C_B)} [M_\bullet]$$

$$= v^{\langle A, B \rangle - \langle \hat{A}, \hat{P}_B \rangle + \langle \hat{P}_A, \hat{B} \rangle} \sum_{M_\bullet \in \mathcal{X}} \frac{|\mathrm{Ext}^1_{\mathcal{C}(\mathcal{R})}(C_A, C_B)_{M_\bullet}|}{|\mathrm{Hom}_{\mathcal{R}}(A, B)|} [M_\bullet].$$

It follows by (3.14) that

$$E_A \circledast E_B = v^{\langle \hat{P}_A, \hat{A} \rangle + \langle \hat{P}_B, \hat{B} \rangle + (\hat{P}_B, \hat{A})} \cdot K_{-P_A \hat{\oplus} P_B} \circledast [C_A] \circledast [C_B]$$

$$= v^{\langle A, B \rangle + \langle \hat{P}_A + \hat{P}_B, \hat{A} + \hat{B} \rangle} \sum_{M_\bullet \in \mathcal{X}} \frac{|\mathrm{Ext}^1_{\mathcal{C}(\mathcal{R})}(C_A, C_B)_{M_\bullet}|}{|\mathrm{Hom}_{\mathcal{R}}(A, B)|} K_{-P_A \hat{\oplus} P_B} \circledast [M_\bullet]. \tag{4.1}$$

Consider an extension

$$0 \longrightarrow C_B \longrightarrow M_\bullet \longrightarrow C_A \longrightarrow 0. \tag{4.2}$$

By Lemma 3.7, we may assume $M_\bullet = \mathrm{Cone}(s_\bullet)$ for some morphism $s_\bullet : C_A \to C_B^*$, so that

$$M_\bullet = P_B \oplus P_A \underset{d_0}{\overset{d_1}{\rightleftarrows}} Q_B \oplus Q_A,$$

where

$$d_1 = \begin{pmatrix} f_B & s_1 \\ 0 & f_A \end{pmatrix}, \qquad d_0 = \begin{pmatrix} 0 & s_0 \\ 0 & 0 \end{pmatrix}.$$

Since f_A, f_B are monomorphisms, so is d_1. Thus $d_1 \circ d_0 = 0$ implies that $s_0 = 0$. Setting $C = H_0(M_\bullet)$, it follows that (4.2) induces an extension

$$0 \longrightarrow B \longrightarrow C \longrightarrow A \longrightarrow 0.$$

One may check that this extension agrees with the corresponding image of (4.2) under the isomorphism

$$\mathrm{Ext}^1_{\mathcal{C}(\mathcal{R})}(C_A, C_B) \cong \mathrm{Ext}^1_{\mathcal{R}}(A, B)$$

given by Lemma 3.7 and Lemma 3.8 (ii). It follows that

$$\sum_{H_0(M_\bullet) = C} |\mathrm{Ext}^1_{\mathcal{C}(\mathcal{R})}(C_A, C_B)_{M_\bullet}| \;=\; |\mathrm{Ext}^1_{\mathcal{R}}(A, B)_C|.$$

Finally, notice that $\hat{C} = \hat{A} + \hat{B}$ for any extension C of A by B. Putting everything together shows that equation (4.1) becomes

$$E_A \circledast E_B = v^{\langle A,B\rangle} \sum_{M_\bullet \in \mathcal{X}} v^{\langle \hat{P}_A + \hat{P}_B,\, \hat{H}_0(M_\bullet)\rangle} \cdot \frac{|\mathrm{Ext}^1_{\mathcal{C}(\mathcal{R})}(C_A, C_B)_{M_\bullet}|}{|\mathrm{Hom}_{\mathcal{R}}(A,B)|} K_{-P_A \hat{\oplus} P_B} \circledast [M_\bullet]$$

$$= v^{\langle A,B\rangle} \sum_{C \in \mathrm{Iso}(\mathcal{R})} \frac{|\mathrm{Ext}^1_{\mathcal{R}}(A,B)_C|}{|\mathrm{Hom}_{\mathcal{R}}(A,B)|} E_C$$

which completes the proof. □

Lemma 4.2. *Let $A, B \in \mathcal{A}$. The following equations hold in $\mathcal{DH}(\mathcal{R})$,*

(i) $E_A \circledast F_B = \sum_{A_1,B_1,B_2} v^{\langle \hat{B}-\hat{B}_1,\, \hat{A}-\hat{B}\rangle} g^B_{B_1,B_2} g^A_{B_2,A_1} a_{B_2} \cdot K^*_{\hat{B}-\hat{B}_1} \circledast E_{A_1,B_1},$

(ii) $F_B \circledast E_A = \sum_{A_1,A_2,B_1} v^{\langle \hat{A}-\hat{A}_1,\, \hat{B}-\hat{A}\rangle} g^A_{A_1,A_2} g^B_{A_2,B_1} a_{A_2} \cdot K_{\hat{A}-\hat{A}_1} \circledast E_{A_1,B_1},$

where each sum runs over classes of objects in $\mathrm{Iso}(\mathcal{A})$.

Proof. (i) Again let C_A, C_B be complexes associated to A, B as in Definition 3.16. By definition, the product $[C_A] \circledast [C^*_B]$ is equal to

$$v^{\langle \hat{Q}_A, \hat{P}_B\rangle + \langle \hat{P}_A, \hat{Q}_B\rangle} \sum_{M_\bullet \in \mathcal{X}} \frac{\mathrm{e}(C_A, C^*_B)_{M_\bullet}}{\mathrm{h}(C_A, C^*_B)} [M_\bullet].$$

Then using $\mu(C_A, C^*_B) = \langle \hat{P}_A, \hat{Q}_B\rangle$, it is easy to check that

$$\mathrm{h}(C_A, C^*_B) = q^{\langle \hat{P}_A, \hat{Q}_B\rangle}.$$

This gives

$$[C_A] \circledast [C^*_B] = v^{\langle \hat{A}, \hat{P}_B\rangle - \langle \hat{P}_A, \hat{B}\rangle} \sum_{M_\bullet \in \mathcal{X}} \mathrm{e}(C_A, C^*_B)_{M_\bullet} \cdot [M_\bullet],$$

where we have used the equalities $\hat{Q}_A = \hat{P}_A + \hat{A}$ and $\hat{Q}_B = \hat{P}_B + \hat{B}$ in $K(\mathcal{R})$. It thus follows by (3.14) that

$$E_A \circledast F_B = v^{\langle \hat{P}_A, \hat{A}\rangle + \langle \hat{P}_B, \hat{B}\rangle - \langle \hat{P}_B, \hat{A}\rangle} \cdot K_{-\hat{P}_A} \circledast K^*_{-\hat{P}_B} \circledast [C_A] \circledast [C^*_B]$$

$$= v^{\langle \hat{P}_A - \hat{P}_B, \hat{A}-\hat{B}\rangle} \cdot K_{-\hat{P}_A} \circledast K^*_{-\hat{P}_B} \circledast \sum_{M_\bullet \in \mathcal{X}} \mathrm{e}(C_A, C^*_B)_{M_\bullet} \cdot [M_\bullet]. \tag{4.3}$$

Now suppose that $M_\bullet$ is an extension of C_A by C_B^*. By Lemma 3.7, we may assume that $M_\bullet = \mathrm{Cone}(s_\bullet)$ for some $s_\bullet \in \mathrm{Hom}_{\mathrm{Ho}(\mathcal{R})}(C_A, C_B)$. The extension thus takes the form

$$\begin{tikzcd}
Q_B \arrow[r, shift left, "0"] \arrow[d, "i_1"'] & P_B \arrow[l, shift left, "-f_B"] \arrow[d, "i_0"] \\
Q_B \oplus P_A \arrow[r, shift left, "f_1"] \arrow[d, "p_1"'] & P_B \oplus Q_A \arrow[l, shift left, "f_0"] \arrow[d, "p_0"] \\
P_A \arrow[r, shift left, "f_A"] & Q_A \arrow[l, shift left, "0"]
\end{tikzcd}$$

where $f_0 = \left(\begin{smallmatrix} -f_B & s_0 \\ 0 & 0 \end{smallmatrix}\right)$, $f_1 = \begin{pmatrix} 0 & s_1 \\ 0 & f_A \end{pmatrix}$, and f_A, f_B are defined in (3.16). This extension induces an exact commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & H_0(M_\bullet) \arrow[r] \arrow[d] & H_0(C_A) \arrow[d] & \\
 & P_B \arrow[r, "i_0"] \arrow[d, "-f_B"'] & M_0/\operatorname{im} f_1 \arrow[r, "p_0"] \arrow[d, "f_0"] & Q_A/\operatorname{im} f_A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Q_B \arrow[r, "i_1"] \arrow[d] & \ker f_1 \arrow[r, "p_1"] \arrow[d] & 0 \arrow[d] & \\
 & H_1(C_B^*) \arrow[r] & H_1(M_\bullet) \arrow[r] & 0, &
\end{tikzcd} \tag{4.4}$$

where "i_0" denotes the map induced by i_0, etc.

Since the map induced by i_1 in (4.4) is an isomorphism, it follows that the direct summands in the decomposition $M_\bullet = M_\bullet^+ \oplus M_\bullet^-$ of Lemma 3.2 have the form

$$M_\bullet^+ = (P_A \underset{0}{\overset{f_1^+}{\rightleftarrows}} M_0^+), \quad M_\bullet^- = (Q_B \underset{f_0^-}{\overset{0}{\rightleftarrows}} M_0^-)$$

where the maps f_0^-, f_1^+ are obtained from f_0, f_1 by restriction.

The objects $A_1 = H_0(M_\bullet)$ and $B_1 = H_1(M_\bullet)$ thus have projective resolutions

$$P_A \xrightarrow{f_1^+} M_0^+ \longrightarrow A_1 \longrightarrow 0, \qquad M_0^- \xrightarrow{f_0^-} Q_B \longrightarrow B_1 \longrightarrow 0$$

respectively. This gives relations

$$\hat{M}_1^+ = \hat{P}_A, \qquad \hat{M}_0^- = \hat{Q}_B - \hat{B}_1 = \hat{P}_B + \hat{B} - \hat{B}_1$$

in $K(\mathcal{R})$. Substituting in (4.3), we have

$$\begin{aligned}
E_A \circledast F_B &= \sum_{M_\bullet \in \mathcal{X}} v^{\langle \hat{M}_1^+ - \hat{M}_0^- + \hat{B} - \hat{B}_1, \hat{A} - \hat{B} \rangle} \cdot \mathrm{e}(C_A, C_B^*)_{M_\bullet} \cdot K_{-\hat{M}_0^+} \circledast K^*_{-\hat{M}_0^- + \hat{B} - \hat{B}_1} \circledast [M_\bullet] \\
&= \sum_{M_\bullet \in \mathcal{X}} v^{\langle \hat{B} - \hat{B}_1, \hat{A} - \hat{B} \rangle} \cdot \mathrm{e}(C_A, C_B^*)_{M_\bullet} \cdot K^*_{\hat{B} - \hat{B}_1} \circledast E_{M_\bullet}. \qquad (4.5)
\end{aligned}$$

One may check directly using (4.4) that the map $s : A \to B$ induced by (3.5) coincides with the connecting homomorphism $H_0(C_A) \xrightarrow{\delta} H_1(C_B)$ in the long exact sequence of cohomology. In particular, note that $H_0(M_\bullet) \simeq \ker s$, and $H_1(M_\bullet) \simeq \operatorname{coker} s$.

Hence, we may conclude that

$$\sum_{\substack{P_\bullet \in \mathcal{X} \\ H_0(P_\bullet)\simeq A_1,\, H_1(P_\bullet)\simeq B_1}} \mathrm{e}(C_A, C_B^*)_{P_\bullet} \cdot E_{P_\bullet} = |\{h \in \mathrm{Hom}_{\mathcal{R}}(A,B) \mid \ker h \simeq A_1, \operatorname{coker} h \simeq B_1\}| \cdot E_{A_1,B_1}. \tag{4.6}$$

By the equality on [23, p.984], the preceding equation may be rewritten as

$$\sum_{\substack{H_0(P_\bullet)\simeq A_1, \\ H_1(P_\bullet)\simeq B_1}} \mathrm{e}(C_A, C_B^*)_{P_\bullet} \cdot E_{P_\bullet} = \sum_{B_2 \in \mathrm{Iso}(\mathcal{A})} g^B_{B_1,B_2} g^A_{B_2,A_1} a_{B_2} \cdot E_{A_1,B_1}. \tag{4.7}$$

The equality in part (i) is now obtained by combining (4.3), (4.7) and (2.3).

(ii) This follows by interchanging A and B in (i) and taking $*$ on both sides. □

4.2. **Embedding $\widetilde{\mathcal{H}_v}(\mathcal{R})$ in $\mathcal{DH}(\mathcal{R})$.** In this subsection we make some more precise statements about the relationships between the various Hall algebras we have been considering.

Consider the injective linear map $I_+ \colon \widetilde{\mathcal{H}_v}(\mathcal{R}) \hookrightarrow \mathcal{DH}(\mathcal{R})$ defined by

$$[A] * K_\alpha \mapsto E_A \circledast K_\alpha,$$

and let $\mathcal{DH}^+(\mathcal{R}) \subset \mathcal{DH}(\mathcal{R})$ denote the image of this map.

Proposition 4.3. *The restriction of multiplication in $\mathcal{DH}(\mathcal{R})$ makes the subspace $\mathcal{DH}^+(\mathcal{R})$ into an associative algebra, and the embedding $I_+ \colon \widetilde{\mathcal{H}_v}(\mathcal{R}) \hookrightarrow \mathcal{DH}(\mathcal{R})$ restricts to an isomorphism $\widetilde{\mathcal{H}_v}(\mathcal{R}) \cong \mathcal{DH}^+(\mathcal{R})$ of (unital) associative algebras.*

Proof. The result follows from Lemma 4.1, together with a comparison of the relations (2.6) defining the extended Hall algebra with the relation (3.14) in the localized Hall algebra. □

Composing with the involution $*$ gives another embedding

$$I_- \colon \widetilde{\mathcal{H}_v}(\mathcal{R}) \hookrightarrow \mathcal{DH}(\mathcal{R}),$$

defined by $[B] * K_\beta \mapsto F_B \circledast K_\beta^*$, whose image $\mathcal{DH}^-(\mathcal{R})$ is again an associative algebra such that I_- restricts to an algebra isomorphism $\widetilde{\mathcal{H}_v}(\mathcal{R}) \cong \mathcal{DH}^-(\mathcal{R})$.

4.3. **Drinfeld double of $\widetilde{\mathcal{H}_v}(\mathcal{R})$.**

Lemma 4.4. *The multiplication map $\nabla \colon a \otimes b \mapsto I_+(a) \circledast I_-(b)$ defines an isomorphism of vector spaces*

$$\nabla \colon \widetilde{\mathcal{H}_v}(\mathcal{R}) \otimes_{\mathbb{C}} \widetilde{\mathcal{H}_v}(\mathcal{R}) \longrightarrow \mathcal{DH}(\mathcal{R}).$$

Proof. It follows from Proposition 3.18 that the algebra $\mathcal{DH}(\mathcal{R})$ has a $\mathbb{C}$-basis consisting of elements

$$E_{A,B} \circledast K_\alpha \circledast K_\beta^*, \quad A, B \in \mathrm{Iso}(\mathcal{A}), \quad \alpha, \beta \in K(\mathcal{R}).$$

Recall the partial order on $K(\mathcal{A})$ defined in (2.9) and define $\mathcal{DH}_{\leq\gamma}$ for $\gamma \in K(\mathcal{A})$ to be the subspace of $\mathcal{DH}(\mathcal{R})$ spanned by elements from this basis for which $\boldsymbol{k}_{\mathcal{A}}(A) + \boldsymbol{k}_{\mathcal{A}}(B) \leq \gamma$. We claim that

$$\mathcal{DH}_{\leq\gamma} \circledast \mathcal{DH}_{\leq\delta} \subset \mathcal{DH}_{\leq\gamma+\delta}, \qquad \gamma, \delta \in K(\mathcal{A}), \tag{4.8}$$

so that this defines a filtration on $\mathcal{DH}(\mathcal{R})$.

Suppose that $M_\bullet, N_\bullet \in \mathcal{C}_{\mathsf{fin}}(\mathcal{P})$ and let

$$\gamma = \boldsymbol{k}_{\mathcal{A}}(H_0(M_\bullet)) + \boldsymbol{k}_{\mathcal{A}}(H_1(M_\bullet)), \quad \delta = \boldsymbol{k}_{\mathcal{A}}(H_0(N_\bullet)) + \boldsymbol{k}_{\mathcal{A}}(H_1(N_\bullet)).$$

Then consider an extension of complexes

$$0 \longrightarrow M_\bullet \longrightarrow P_\bullet \longrightarrow N_\bullet \longrightarrow 0.$$

The long exact sequence in homology can be split to give two long exact sequences

$$0 \longrightarrow K \longrightarrow H_0(M_\bullet) \longrightarrow H_0(P_\bullet) \longrightarrow H_0(N_\bullet) \longrightarrow L \longrightarrow 0,$$
$$0 \longrightarrow L \longrightarrow H_1(M_\bullet) \longrightarrow H_1(P_\bullet) \longrightarrow H_1(N_\bullet) \longrightarrow K \longrightarrow 0$$

for some objects $K, L \in \mathcal{A}$. It follows that there is a relation in $K(\mathcal{A})$,

$$\gamma + \delta = \boldsymbol{k}_{\mathcal{A}}(H_0(P_\bullet)) + \boldsymbol{k}_{\mathcal{A}}(H_1(P_\bullet)) + 2(\boldsymbol{k}_{\mathcal{A}}(K) + \boldsymbol{k}_{\mathcal{A}}(L))$$

which proves (4.8).

Suppose now that $N_\bullet = C_A$ and $M_\bullet = C_B^*$ for some objects $A, B \in \mathcal{A}$. Then $K = 0$, and by Lemmas 3.1, 3.7, and 3.8

$$\operatorname{Ext}^1_{\mathcal{C}(\mathcal{R})}(N_\bullet, M_\bullet) = \operatorname{Hom}_{\mathcal{R}}(A, B),$$

and the extension class is completely determined by the connecting morphism $H_0(N_\bullet) \to H_1(M_\bullet)$. By Proposition 2.4, we therefore know that $\boldsymbol{k}_{\mathcal{A}}(L) = 0$ exactly when the extension is trivial. It follows that in the graded algebra associated to the filtered algebra $\mathcal{DH}(\mathcal{R})$, one has a relation

$$\nabla([A] * K_\alpha \otimes [B] * K_\beta) = v^{-(\alpha, \hat{B})} \cdot E_{A,B} \circledast K_\alpha \circledast K_\beta^*.$$

It follows that ∇ takes a basis to a basis and is hence an isomorphism. $\square$

As a corollary, we have

Corollary 4.5. *The algebra $\mathcal{DH}(\mathcal{R})$ has a linear basis consisting of elements*

$$E_A \circledast K_\alpha \circledast F_B \circledast K_\beta^*, \quad A, B \in \operatorname{Iso}(\mathcal{A}), \ \alpha, \beta \in K(\mathcal{R}).$$

Now we state the main result of this section.

Theorem 4.6. *Suppose that $\mathcal{A} \subset \mathcal{R}$ satisfies conditions* (2.12) *and* (2.13). *Then the algebra $\mathcal{DH}(\mathcal{R})$ is isomorphic to the Drinfeld double of the bialgebra $\widetilde{\mathcal{H}_v}(\mathcal{R})$.*

Proof. Because of the description of the basis of $\mathcal{DH}(\mathcal{R})$ (Corollary 4.5) and the definition of Drinfeld double, the proof of the theorem is reduced to check equation (2.14) for the elements consisting of the basis of $\widetilde{\mathcal{H}_v}(\mathcal{R})$.

Let us write equation (2.14) in the present situation:

$$\sum (a_2, b_1)_H \cdot I_+(a_1) \circledast I_-(b_2) \overset{?}{=} \sum (a_1, b_2)_H \cdot I_-(b_1) \circledast I_+(a_2). \tag{4.9}$$

Now let $A, B \in \mathcal{A}$ and $\alpha, \beta \in K(\mathcal{R})$. Let us write

$$\Delta([A]K_\alpha) = \sum_{A_1,A_2} v^{\langle A_1,A_2\rangle} g^A_{A_1,A_2} \cdot ([A_1]K_{\hat{A}_2+\alpha}) \otimes ([A_2]K_\alpha),$$
$$\Delta([B]K_\beta) = \sum_{B_2,B_1} v^{\langle B_2,B_1\rangle} g^B_{B_2,B_1} \cdot ([B_2]K_{\hat{B}_1+\beta}) \otimes ([B_1]K_\beta).$$

By the Hopf pairing (Definition 2.6 and Proposition 2.7) and (3.14), the left hand side of (4.9) becomes

$$\begin{aligned}\text{LHS of (4.9)} = \sum_{A_1,A_2,B_1,B_2} & v^{\langle A_1,A_2\rangle} g^A_{A_1,A_2} v^{\langle B_2,B_1\rangle} g^B_{B_2,B_1} ([A_2]K_\alpha, [B_2]K_{\hat{B}_1+\beta})_H \\ & \cdot E_{A_1} \circledast K_{\hat{A}_2+\alpha} \circledast F_{B_1} \circledast K^*_\beta \\ = v^{(\alpha,\beta)} \sum_{A_1,A_2,B_1,B_2} & v^{\langle A_1,A_2\rangle+\langle B_2,B_1\rangle} g^A_{A_1,A_2} g^B_{B_2,B_1} ([A_2],[B_2])_H \\ & \cdot E_{A_1} \circledast K_{\hat{A}_2} \circledast F_{B_1} \circledast K_\alpha \circledast K^*_\beta.\end{aligned}$$

Similarly, the right hand side becomes

$$\begin{aligned}&\text{RHS of (4.9)} \\ &= v^{(\alpha,\beta)} \sum_{A_1,A_2,B_1,B_2} v^{\langle A_1,A_2\rangle+\langle B_2,B_1\rangle} g^A_{A_1,A_2} g^B_{B_2,B_1} ([A_1],[B_1])_H \cdot F_{B_2} \circledast K^*_{\hat{B}_1} \circledast E_{A_2} \circledast K_\alpha \circledast K^*_\beta.\end{aligned}$$

After removing the term $v^{(\alpha,\beta)} \cdot K_\alpha \circledast K^*_\beta$ from both sides, equation (4.9) reduces to

$$\begin{aligned}&\sum_{A_1,A_2,B_1,B_2} v^{\langle A_1,A_2\rangle+\langle B_2,B_1\rangle} g^A_{A_1,A_2} g^B_{B_2,B_1} ([A_2],[B_2])_H \cdot E_{A_1} \circledast K_{\hat{A}_2} \circledast F_{B_1} \\ &\quad \stackrel{?}{=} \sum_{A_1,A_2,B_1,B_2} v^{\langle A_1,A_2\rangle+\langle B_2,B_1\rangle} g^A_{A_1,A_2} g^B_{B_2,B_1} ([A_1],[B_1])_H \cdot F_{B_2} \circledast K^*_{\hat{B}_1} \circledast E_{A_2}.\end{aligned} \tag{4.10}$$

By Definition 2.6 and (3.14), the left hand side of (4.10) becomes

$$\begin{aligned}\text{LHS of (4.10)} &= \sum_{A_1,A_2,B_1,B_2} v^{\langle A_1,A_2\rangle+\langle B_2,B_1\rangle} g^A_{A_1,A_2} g^B_{B_2,B_1} a_{A_2} \delta_{A_2,B_2} \cdot E_{A_1} \circledast K_{\hat{A}_2} \circledast F_{B_1} \\ &= \sum_{A_1,A_2,B_1} v^{\langle A_1,A_2\rangle+\langle A_2,B_1\rangle} g^A_{A_1,A_2} g^B_{A_2,B_1} a_{A_2} \cdot E_{A_1} \circledast K_{\hat{A}_2} \circledast F_{B_1} \\ &= \sum_{A_1,A_2,B_1} v^{\langle \hat{A}_2,\hat{B}_1\rangle-\langle \hat{A}_2,\hat{A}_1\rangle} g^A_{A_1,A_2} g^B_{A_2,B_1} a_{A_2} \cdot K_{\hat{A}_2} \circledast E_{A_1} \circledast F_{B_1}.\end{aligned}$$

Thus by Lemma 4.2 (i) we have

$$\begin{aligned}\text{LHS of (4.10)} = \sum_{A_1,A_2,A_3,B_1,B_2,B_3} & v^{\langle \hat{B}_2-\hat{A}_2,\, \hat{A}-\hat{B}\rangle} g^{A_1}_{B_2,A_3} g^A_{A_1,A_2} g^B_{A_2,B_1} g^{B_1}_{B_3,B_2} \cdot a_{A_2} a_{B_2} \\ & \cdot K_{\hat{A}_2} \circledast K^*_{\hat{B}_2} \circledast E_{A_3,B_3}.\end{aligned}$$

Similar computations using Lemma 4.2 (ii) show that the right hand side of (4.10) becomes

$$\text{RHS of (4.10)} = \sum_{A_1,A_2,A_3,B_1,B_2,B_3} v^{\langle \hat{A}_1-\hat{B}_1,\, \hat{B}-\hat{A}\rangle} g^{A_2}_{A_3,A_1} g^{B_2}_{A_1,B_3} g^{B}_{B_2,B_1} g^{A}_{B_1,A_2} \cdot a_{B_1} a_{A_1} \cdot K_{\hat{A}_1} \circledast K^*_{\hat{B}_1} \circledast E_{A_3,B_3}.$$

It follows by associativity (2.3) that this can be rewritten as

$$\text{RHS of (4.10)} = \sum_{A_1,A_3,B_1,B_3} v^{\langle \hat{A}_1-\hat{B}_1,\, \hat{B}-\hat{A}\rangle} \sum_{A_2',B_2'} g^{A_2'}_{B_1,A_3} g^{A}_{A_2',A_1} g^{B}_{A_1,B_2'} g^{B_2'}_{B_3,B_1} \cdot a_{B_1} a_{A_1} \cdot K_{\hat{A}_1} \circledast K^*_{\hat{B}_1} \circledast E_{A_3,B_3}.$$

One may check that this expression agrees with the LHS of (4.10), which completes the proof. □

It is now possible to verify that the multiplication in $\mathcal{DH}(\mathcal{R})$ is associative.

Corollary 4.7. *If the category $\mathcal{R}$ is Hom-finite (so that $\mathcal{A} = \mathcal{R}$) or if the subcategory $\mathcal{A} \subset \mathcal{R}$ satisfies conditions* (2.12) *and* (2.13)*, then the algebra $\mathcal{DH}(\mathcal{R})$ is associative.*

Proof. If $\mathcal{R} = \mathcal{A}$, then it follows by Remark 3.11 that the algebra $\mathcal{DH}(\mathcal{R})$ is isomorphic to the localized Hall algebra $\mathcal{DH}(\mathcal{A})$ defined in [4]. It follows by results in [4] that the category $\mathcal{C}(\mathcal{A})$ is Hom-finite, so that $\mathcal{H}(\mathcal{C}(\mathcal{P}))$ and $\mathcal{DH}(\mathcal{A})$ are both associative in this case.

The remaining statement is a consequence of Theorem 4.6 since the multiplication in the Drinfeld double is associative. □

Remark 4.8. From the above result we can only conclude that the algebra $\mathcal{H}(\mathcal{C}_{\mathsf{fin}}(\mathcal{P}))$ is "locally associative" in general, in the sense that given $x, y, z \in \mathcal{H}(\mathcal{C}_{\mathsf{fin}}(\mathcal{P}))$ we have

$$u \circledast (x \circledast (y \circledast z) - (x \circledast y) \circledast z) = 0$$

for some element $u \in \mathcal{Z}$.

4.4. **Reduction.** Define the reduced localized Hall algebra by setting $[M_\bullet] = 1$ in $\mathcal{DH}(\mathcal{R})$ whenever $M_\bullet$ is an acyclic complex, invariant under the shift functor. More formally, we set

$$\mathcal{DH}_{\mathsf{red}}(\mathcal{R}) = \mathcal{DH}(\mathcal{R}) / \big([M_\bullet] - 1 : H_\bullet(M_\bullet) = 0,\ M_\bullet \cong M_\bullet^*\big).$$

By Lemma 3.4 this is the same as setting

$$[K_P] \circledast [K_P^*] = 1$$

for all $P \in \mathcal{P}$. One can check that the shift functor $*$ defines involutions of $\mathcal{DH}_{\mathsf{red}}(\mathcal{R})$.

We have the following triangular decomposition.

Proposition 4.9. *The multiplication map $[A] \otimes \alpha \otimes [B] \mapsto E_A \circledast K_\alpha \circledast F_B$ defines an isomorphism of vector spaces*

$$\mathcal{H}(\mathcal{A}) \otimes_{\mathbb{C}} \mathbb{C}[K(\mathcal{R})] \otimes_{\mathbb{C}} \mathcal{H}(\mathcal{A}) \longrightarrow \mathcal{DH}_{\mathsf{red}}(\mathcal{R}).$$

Proof. The same argument given for Lemma 4.4 also applies here. □

4.5. **Commutation relations.** In this subsection, we prove commutation relations among generators that are important to understand $\mathcal{DH}_{\mathsf{red}}(\mathcal{R})$.

Lemma 4.10. *Suppose* $A_1, A_2 \in \mathcal{A}$ *satisfy*

$$\mathrm{Hom}_{\mathcal{R}}(A_1, A_2) = 0 = \mathrm{Hom}_{\mathcal{R}}(A_2, A_1).$$

Then $[E_{A_1}, F_{A_2}] = 0$.

Proof. It follows from (4.5) and (4.6) that $E_{A_1} \circledast F_{A_2} = E_{A_1,A_2}$. Exchanging A_1 and A_2 in this equation and taking $*$ on both sides gives $F_{A_2} \circledast E_{A_1} = E_{A_1,A_2}$ as well, and the result follows. $\square$

Lemma 4.11. *Suppose* $A \in \mathcal{A}$ *satisfies* $\mathrm{End}_{\mathcal{A}}(A) = \Bbbk$. *Then*

$$[E_A, F_A] = (q-1) \cdot (K^*_{\hat{A}} - K_{\hat{A}}).$$

Proof. Using formulas (4.5) and (4.6) again, we have $E_A \circledast F_A = E_{A,A} + (q-1) \cdot K^*_A$. Taking $*$ on both sides gives the equation $F_A \circledast E_A = E_{A,A} + (q-1) \cdot K_A$, since $E_{A,A}$ is $*$-invariant. The result follows by subtracting these equations. $\square$

5. Realization of quantum groups

5.1. **Quivers.** Let $\mathcal{Q}$ be a locally finite quiver with vertex set I and (oriented) edge set Ω. For $\sigma \in \Omega$ we denote by $h(\sigma)$ and $t(\sigma)$ the head and tail, respectively, and sometimes use the notation $t(\sigma) \xrightarrow{\sigma} h(\sigma)$. We will denote by c_i the number of loops at $i \in I$ (i.e., the number of edges σ with $h(\sigma) = t(\sigma) = i$). A (finite) *path* in $\mathcal{Q}$ is a sequence $\sigma_m \cdots \sigma_1$ of edges which satisfies $h(\sigma_i) = t(\sigma_{i+1})$ for $1 \le i < m$. For each $i \in I$, we let e_i denote the trivial path. We again let $h(x)$ and $t(x)$ denote the head and tail vertices of a path x.

Consider the sub-quiver $\bar{\mathcal{Q}} \subset \mathcal{Q}$ with vertex set $\bar{I} = I$ and edge set $\bar{\Omega} = \Omega \backslash \{\sigma | h(\sigma) = t(\sigma)\}$. We make the following assumptions throughout:

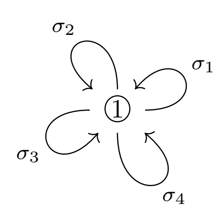
(A) There are no infinite paths of the form $i_0 \to i_1 \to i_2 \to \cdots$ in $\bar{\mathcal{Q}}$. In particular, $\bar{\mathcal{Q}}$ is acyclic and $\mathcal{Q}$ has no oriented cycles other than loops.
(B) Each vertex of the quiver $\mathcal{Q}$ has either zero loops or more than one loop, i.e. $c_i \neq 1$ for all $i \in I$.

It follows from (A) that I is partially ordered, with $i \preceq j$ if there exists a path x such that $t(x) = j$ and $h(x) = i$, and the set $(I, \preceq)$ satisfies the descending chain condition.

From now on, we assume that the quiver $\mathcal{Q}$ satisfies the conditions (A) and (B).

Example 5.1. Write $\mathbb{L}_n$ to denote the quiver consisting of a single vertex $I = \{1\}$ and n loops. If $n \geq 2$, then the quiver $\mathcal{Q} = \mathbb{L}_n$ trivially satisfies the assumptions (A) and (B). Below is a diagram for $\mathbb{L}_4$.

$\mathbb{L}_4$:



Let $R = \Bbbk\mathcal{Q}$ denote the path algebra, with basis given by the set of paths and multiplication defined via concatenation. The elements e_i are pairwise orthogonal idempotents. It follows from (A) that the subring e_iRe_i is isomorphic to a free associative $\Bbbk$-algebra on c_i generators. (If $c_i = 0$ then $e_iRe_i \cong \Bbbk$.) Hence each left ideal $P_i := Re_i$ is an indecomposable projective R-module. It can then be checked that R splits as a direct sum

$$R = \bigoplus_{i \in I} P_i \tag{5.1}$$

of pairwise non-isomorphic projective left R-submodules.

A representation of $\mathcal{Q}$ over $\Bbbk$ is a collection $(V_i, x_\sigma)_{i\in I,\sigma\in\Omega}$, where V_i is a (possibly infinite dimensional) $\Bbbk$-vector space and $x_\sigma \in \operatorname{Hom}_\Bbbk(V_{t(\sigma)}, V_{h(\sigma)})$. We let $\operatorname{Rep}_\Bbbk(\mathcal{Q})$ denote the abelian category consisting of the representations of $\mathcal{Q}$ over $\Bbbk$ which are of *finite support*, i.e. such that $V_i = 0$ for almost all i. A representation $(V_i, x_\sigma) \in \operatorname{Rep}_\Bbbk(\mathcal{Q})$ is called *finite-dimensional* if each V_i is finite-dimensional. For such a representation, set $\mathbf{dim}(V_i, x_\sigma) = \mathbf{dim}(V_i) = (\dim_\Bbbk V_i) \in \mathbb{N}^{\oplus I}$. We denote by $\operatorname{rep}_\Bbbk(\mathcal{Q})$ the full subcategory of $\mathcal{R}$ consisting of the finite dimensional representations of $\mathcal{Q}$.

Any representation of $\mathcal{Q}$ is naturally an R-module for the path algebra R. Whenever it is convenient, particularly in the next subsection, we will consider representations of $\mathcal{Q}$ as R-modules. It follows from the decomposition (5.1) that any left R-module M has a decomposition into $\Bbbk$-subspaces, $M = \bigoplus_{i\in I} M(i)$, with $M(i) = e_iM$. Then we have $\mathbf{dim}(M) = (\dim_\Bbbk M(i)) \in \mathbb{N}^{\oplus I}$ for a finite dimensional R-module M, which is equal to the dimension vector as a representation. We say that an R-module is of finite support if the associated representation is.

Recall that any $M \in \operatorname{Rep}_\Bbbk(\mathcal{Q})$ has the standard presentation of the form

$$0 \longrightarrow \bigoplus_{\sigma\in\Omega} P_{h(\sigma)} \otimes_\Bbbk e_{t(\sigma)}M \xrightarrow{f} \bigoplus_{i\in I} P_i \otimes_\Bbbk e_iM \xrightarrow{g} M \longrightarrow 0. \tag{5.2}$$

Let $\operatorname{Proj}_\Bbbk(\mathcal{Q})$ denote the full subcategory of $\operatorname{Rep}_\Bbbk(\mathcal{Q})$ whose objects are finitely-generated, projective R-modules. Let $\mathcal{R}$ be the full subcategory of $\operatorname{Rep}_\Bbbk(\mathcal{Q})$ whose objects are finitely presented representations of $\mathcal{Q}$, i.e. the full subcategory consisting of all objects M for which there exists a presentation

$$P \to Q \to M \to 0$$

for some $P, Q \in \operatorname{Proj}_\Bbbk(\mathcal{Q})$.

As in Section 1.1, define $\mathcal{P} \subset \mathcal{R}$ to be the full subcategory of projectives in $\mathcal{R}$ and $\mathcal{A} \subset \mathcal{R}$ the full subcategory of objects $A \in \mathcal{R}$ such that $\operatorname{Hom}_\mathcal{R}(M, A)$ is a finite set for any $M \in \mathcal{R}$. It is easy to see that $\mathcal{P} = \operatorname{Proj}_\Bbbk(\mathcal{Q})$. For the category $\mathcal{A}$, we have the following characterization.

Lemma 5.2. *The category $\mathcal{A}$ is equal to the full subcategory of $\mathcal{R}$ consisting of all finite-dimensional representations, i.e. $\mathcal{A} = \operatorname{rep}_\Bbbk(\mathcal{Q})$.*

Proof. Assume that $A \in \operatorname{rep}_\Bbbk(\mathcal{Q})$. From the standard resolution (5.2), we see that $A \in \mathcal{R}$. Clearly, A is finite as a set. Since any $M \in \mathcal{R}$ is finitely generated, the set $\operatorname{Hom}_\mathcal{R}(M, A)$ is also finite. Thus A is an object of $\mathcal{A}$. For the converse, assume that $M \in \operatorname{Rep}_\Bbbk(\mathcal{Q})$ is infinite dimensional. Then there is a vertex $i \in I$ such that e_iM is infinite dimensional. For each $a \in e_iM$, we have a homomorphism $P_i \to M$ given by $e_i \mapsto a$. Thus $\operatorname{Hom}_\mathcal{R}(P_i, M)$ is an infinite set and M does not belong to $\mathcal{A}$. $\square$

5.2. **Krull–Schmidt property for** $\mathrm{Proj}_{\Bbbk}(\mathcal{Q})$**.** Since the endomorphism ring $\mathrm{End}_R(P_i)$ is not local in general, the usual Krull–Schmidt theorem does not hold in the category $\mathrm{Proj}_{\Bbbk}(\mathcal{Q})$. In this subsection, we describe a suitable analogue.

First note the following.

Lemma 5.3. *Suppose $i, j \in I$ are distinct vertices such that $(Re_iR) \cap (Re_jR) \neq 0$. Then either $i \prec j$ or $j \prec i$.*

Proof. It follows from the stated condition that there are paths x, x', y, y' such that xe_iy and $x'e_jy'$ are both nonzero and
$$xy = xe_iy = x'e_jy' = x'y'.$$
We then have $xy = x'y' = \sigma_1 \cdots \sigma_n$, for some $\sigma_1, \dots, \sigma_n \in \Omega$. Let $1 \leq l, l' \leq n$ be such that $x = \sigma_1 \cdots \sigma_l$, $y = \sigma_{l+1} \cdots \sigma_n$, $x' = \sigma_1 \cdots \sigma_{l'}$, and $y' = \sigma_{l'+1} \cdots \sigma_n$. Suppose without loss of generality that $l < l'$. Then $z = \sigma_{l+1} \cdots \sigma_{l'}$ is a path such that $h(z) = i$ and $t(z) = j$. Thus $i \prec j$. $\square$

Let R-Mod denote the abelian category of all left R-modules of finite support. We identify R-Mod with $\mathrm{Rep}_{\Bbbk}(\mathcal{Q})$, and consider $\mathcal{P} = \mathrm{Proj}_{\Bbbk}(\mathcal{Q})$ as a full subcategory of R-Mod. We write $\mathrm{supp}(M) := \{i \mid M(i) \neq 0\}$ for $M \in R$-Mod. Recall that for any idempotent $e \in R$ there is an exact functor from the category R-Mod to (eRe)-Mod given by $M \mapsto eM$. Now suppose $M \in \mathcal{P}$. Then $M(i)$ is a finitely-generated, projective (e_iRe_i)-module. Since e_iRe_i is a free associative $\Bbbk$-algebra, $M(i)$ is a free (e_iRe_i)-module of finite rank, say $r_i(M)$. (See, for example, [7].) Write $\mathbf{rk}_{\mathcal{P}}(M) = (r_i(M))_{i \in I} \in \mathbf{N}^{\oplus I}$ to denote the vector formed by these ranks. Notice that $r_i(P_i) = 1$ and $r_j(P_i) = 0$ unless $j \preceq i$. It follows that the vectors, $\mathbf{rk}_{\mathcal{P}}(P_i)$, form a basis for $\mathbb{Z}^{\oplus I}$.

Given a subset $J \subseteq I$, consider the set $J_{\preceq} = \{i \in I \mid i \preceq j \text{ for some } j \in J\}$. If J is finite then so is $J_{\preceq}$ by (A), and there are corresponding idempotents
$$e_J = \sum_{j \in J} e_j \qquad \text{and} \qquad e_{\preceq J} = \sum_{i \in J_{\preceq}} e_i.$$

Lemma 5.4. *Suppose $J \subseteq I$ is a finite subset and write $e = e_{\preceq J}$. Then there is an equivalence between (eRe)-Mod and the full subcategory of R-Mod consisting of modules M such that $\mathrm{supp}(M) \subseteq J$.*

Proof. Consider the functor R-Mod $\to (eRe)$-Mod : $M \mapsto eM$. Then $eRe = Re$, and an inverse functor is given by extending the action of Re on $M \in (eRe)$-Mod to all of R by letting $\bigoplus_{i \notin J_{\preceq}} P_i$ act by zero. $\square$

Lemma 5.5. *Suppose $\mathfrak{a}$ is a finitely generated left ideal of R. For $i, j \in I$, the following hold.*

(i) The ideal $\mathfrak{a}$ is a projective left R-module.

(ii) Any nonzero R-module homomorphism, $\phi : P_i \to \mathfrak{a}$, is injective.

(iii) Suppose $\phi_i : P_i \to \mathfrak{a}$ and $\phi_j : P_j \to \mathfrak{a}$ are homomorphisms such that $\mathrm{im}(\phi_i) \cap \mathrm{im}(\phi_j) \neq 0$. Then either $\mathrm{im}(\phi_i) \subseteq \mathrm{im}(\phi_j)$ or $\mathrm{im}(\phi_j) \subseteq \mathrm{im}(\phi_i)$.

(iv) As a left R-module, $\mathfrak{a}$ is isomorphic to a finite direct sum of copies of the modules $\{P_i\}_{i \in I}$.

Proof. (i) Suppose $\mathfrak{a} \subseteq R$ is a left ideal with $S \subseteq \mathfrak{a}$ a finite set of generators. Then $S \subseteq Re_J$ for some finite set J and it follows that $\mathrm{supp}(\mathfrak{a}) \subseteq J_{\preceq}$. If we set $e = e_{\preceq J}$, then $\mathfrak{a} \subseteq eRe$, which shows that $\mathfrak{a}$ is a left ideal of a hereditary ring and thus projective as an eRe-module. It follows that $\mathfrak{a}$ is a projective R-module by Lemma 5.4.

(ii) The image $\mathrm{im}(\phi) \subseteq \mathfrak{a}$ is a projective left R-module by (i). So the exact sequence

$$0 \to \ker(\phi) \to P_i \to \mathrm{im}(\phi) \to 0$$

splits. Since P_i is indecomposable, it follows that ϕ is injective.

(iii) Letting $\phi_i(e_i) = v$ and $\phi_j(e_j) = w$, we have $\mathrm{im}(\phi_i) = Rv$ and $\mathrm{im}(\phi_j) = Rw$. We thus have $(Re_i v) \cap (Re_j w) \neq 0$. It follows by Lemma 5.3 that $i \preceq j$ or $j \preceq i$. Assume without loss of generality that $j \preceq i$. It then follows from the proof of Lemma 5.3, that there exists a path x such that $w = e_j w = e_j x e_i v = xv$. It follows that $\mathrm{im}(\phi_j) = Rw = R(xv) \subseteq Rv = \mathrm{im}(\phi_i)$.

(iv) First set $J^1 := \mathrm{supp}(\mathfrak{a})$, and choose a maximal vertex $j_1 \in J^1$. Then $\mathfrak{a}_{j_1}$ is an $e_{j_1}Re_{j_1}$-module with a finite set, say $\mathcal{S}^1$, of free generators of size $n_1 := r_{j_1}(\mathfrak{a}) \neq 0$. It follows from (ii) that for each generator $v \in \mathcal{S}^1$ the mapping, $P_{j_1} \to \mathfrak{a} : e_{j_1} \mapsto v$, defines an injective R-module homomorphism. By (iii), we thus have a corresponding isomorphism

$$(P_{j_1})^{\oplus n_1} \xrightarrow{\sim} R\mathcal{S}^1 \subseteq \mathfrak{a}.$$

Next choose a maximal vertex j_2 belonging to the subset

$$J^2 := \{j \in J^1 \mid r_j(\mathfrak{a}) - n_1 \cdot r_j(P_{j_1}) > 0\}.$$

It follows that $\mathfrak{a}_{j_2} = e_{j_2}\mathfrak{a}$ is a free $e_{j_2}Re_{j_2}$-module of rank $n_1 \cdot r_j(P_{j_1}) + n_2$, for some $n_2 > 0$. Let $\mathcal{S}^2 = \mathcal{S}^2_1 \sqcup \mathcal{S}^2_2$ be a set of free $e_{j_2}Re_{j_2}$-generators such that $\mathcal{S}^2_1$ generates $e_{j_2}(R\mathcal{S}^1)$. Then $\mathcal{S}^2_2$ has size n_2, and it follows as in the previous paragraph that we have an embedding

$$(P_{j_2})^{\oplus n_2} \xrightarrow{\sim} R\mathcal{S}^2_2 \subseteq \mathfrak{a}.$$

By the maximality of j_1 we have $j_1 \not\preceq j_2$. It is also clear that $R\mathcal{S}^2_2 \not\subseteq R\mathcal{S}^1$. It follows by (iii) that $(R\mathcal{S}^1) \cap (R\mathcal{S}^2_2) = 0$. We thus obtain an embedding

$$(P_{j_1})^{\oplus n_1} \oplus (P_{j_2})^{\oplus n_2} \xrightarrow{\sim} R\mathcal{S}^1 + R\mathcal{S}^2_2 \subseteq \mathfrak{a}.$$

Continuing in this way the process eventually terminates, since $\mathrm{supp}(\mathfrak{a})$ is finite, yielding the desired decomposition. $\square$

The following is an analogue of the Krull–Schmidt theorem for the category of finitely-generated projective R-modules.

Proposition 5.6. *Given any finitely-generated, projective left R-module $M \in \mathrm{Proj}_{\Bbbk}(\mathcal{Q})$, there is an R-module isomorphism*

$$M \cong \bigoplus_{i \in I} P_i^{\oplus n_i}$$

for some nonnegative integers n_i, only finitely many of which are nonzero. Moreover, given another such decomposition, $M \cong \bigoplus_{i \in I} P_i^{\oplus m_i}$, we must have $m_i = n_i$ for all i.

Proof. Since M is finitely generated, $J = \operatorname{supp}(M)$ is finite. Letting $e = e_{\preceq J}$, we see that M is a projective module of the hereditary ring eRe. It follows from [6, Theorem 5.3] or [14] that M is isomorphic to a finitely generated left ideal of eRe. Hence Lemma 5.5 yields a decomposition

$$M \cong \bigoplus_{i \in I} P_i^{\oplus n_i}.$$

It follows that

$$\mathbf{rk}_{\mathcal{P}}(M) = \sum_{i \in I} n_i \cdot \mathbf{rk}_{\mathcal{P}}(P_i).$$

Since the vectors $\{\mathbf{rk}_{\mathcal{P}}(P_i)\}_{i \in I}$ form a basis for $\mathbb{Z}^{\oplus I}$, the decomposition must be unique. □

5.3. **Assumptions (a)-(e).** Let S_i be a simple module supported only at $i \in I$. Then we obtain from (5.2) the standard resolution

$$0 \longrightarrow P_i' \longrightarrow P_i \longrightarrow S_i \longrightarrow 0,$$

where $P_i' = \bigoplus_{j \in I} P_j^{\oplus n_j}$ for some integers n_j. Then clearly $n_j = 0$ unless $j \preceq i$. In particular, if i is a minimal vertex then $P_i' = P_i^{\oplus c_i}$ and hence

$$(1 - c_i)\, \hat{P}_i = \hat{S}_i \tag{5.3}$$

which is non-zero by assumption (B). Now if $i \in I$ is not minimal, then by assumption (A) the set $\{j \mid j \preceq i\}$ is finite. We may thus use (5.2) and (5.3) inductively to write $\hat{P}_i$ as a linear combination

$$\hat{P}_i = \frac{1}{1 - c_i}\, \hat{S}_i + \sum_{j \prec i} r_{ij}\, \hat{S}_j, \qquad r_{ij} \in \mathbb{Q}. \tag{5.4}$$

The following proposition makes it possible to apply the results in the general setting of the previous sections to the category $\mathcal{R}$ of finitely-presented quiver representations.

Proposition 5.7. *The triple $(\mathcal{R}, \mathcal{P}, \mathcal{A})$ satisfies the assumptions (a)-(e) in Section 1.1.*

Proof. (a) It is clear. (b) Since the path algebra R is hereditary, the category $\mathcal{R}$ is hereditary as well. Furthermore, $\mathcal{R}$ has enough projectives by definition. (c) It is clear that $\mathcal{P} = \operatorname{Proj}_{\Bbbk}(\mathcal{Q})$, so this condition follows easily from Proposition 5.6. (d) It follows from the expression (5.4). (e) If $\boldsymbol{k}_{\mathcal{R}}(A) = \boldsymbol{k}_{\mathcal{R}}(B)$ for $A, B \in \mathcal{A}$, the standard resolution (5.2) tells us that e_iA and e_iB have the same number of elements for each $i \in I$. Then $|\operatorname{Hom}(P_i, A)| = |\operatorname{Hom}(P_i, B)|$ for each $i \in I$, and thus $|\operatorname{Hom}(P, A)| = |\operatorname{Hom}(P, B)|$ for $P \in \mathcal{P}$ by Proposition 5.6. □

It is also clear that the subcategory $\mathcal{A} \subset \mathcal{R}$ satisfies the finite subobjects condition (2.11), since each object $A \in \mathcal{A}$ is a finite dimensional vector space by Lemma 5.2. Thus $\mathcal{A}$ also satisfies conditions (2.12) and (2.13).

Remark 5.8. The quiver $\mathbb{L}_1$ is called the *Jordan quiver*, and its path algebra is isomorphic to the polynomial algebra $\Bbbk[x]$. For each $\lambda \in \Bbbk$, there is the one-dimensional simple modules S_λ over $\Bbbk[x]$ where x acts as λ. Considering the standard resolution (5.2), we see $\widehat{S}_\lambda = 0$, and the assumptions (d) and (e) are not satisfied. Nonetheless, the Jordan quiver is related to classical examples of Hall algebras. One can find details, for example, in [21].

5.4. **Quantum generalized Kac-Moody algebras.** In this subsection we recall the basic definitions concerning quantum generalized Kac–Moody algebras. We keep the assumptions on the choice of v as in Section 1.2.

Let I be a countable index set, and fix a symmetric *Borcherds–Cartan* matrix $A=(a_{ij})_{i,j\in I}$ whose entries a_{ij}, by definition, satisfy (i) $a_{ii}\in\{2,0,-2,-4,\ldots\}$ and (ii) $a_{ij}=a_{ji}\in\mathbb{Z}_{\leq 0}$ for all i,j. Put $I^{re}=\{i\in I\mid a_{ii}=2\}$ and $I^{im}=I\backslash I^{re}$, and assume that we are given a collection of positive integers $\mathbf{m}=(m_i)_{i\in I}$, called the *charge* of A, with $m_i=1$ whenever $i\in I^{re}$. We put

$$[n]=\frac{v^n-v^{-n}}{v-v^{-1}},\qquad [n]!=[1][2]\cdots[n],\qquad \begin{bmatrix} n\\ k\end{bmatrix}=\frac{[n]!}{[n-k]![k]!}.$$

The *quantum generalized Kac–Moody algebra* associated with $(A,\mathbf{m})$ is defined to be the (unital) $\mathbb{C}$-algebra $\mathbf{U}_v$ generated by the elements $K_i, K_i^{-1}, E_{ik}, F_{ik}$ for $i\in I$, $k=1,\ldots,m_i$, subject to the following set of relations: for $i,j\in I$, $k=1,\ldots,m_i$ and $l=1,\ldots,m_j$,

$$K_iK_i^{-1}=K_i^{-1}K_i=1,\quad K_iK_j=K_jK_i, \tag{5.5}$$

$$K_iE_{jl}K_i^{-1}=v^{a_{ij}}E_{jl},\qquad K_iF_{jl}K_i^{-1}=v^{-a_{ij}}F_{jl}, \tag{5.6}$$

$$E_{ik}F_{jl}-F_{jl}E_{ik}=\delta_{lk}\delta_{ij}\frac{K_i-K_i^{-1}}{v-v^{-1}}, \tag{5.7}$$

$$E_{ik}E_{jl}-E_{jl}E_{ik}=F_{ik}F_{jl}-F_{jl}F_{ik}=0\qquad \text{if } a_{ij}=0,\quad\text{and} \tag{5.8}$$

$$\begin{aligned}&\sum_{n=0}^{1-a_{ij}}(-1)^n\begin{bmatrix}1-a_{ij}\\ n\end{bmatrix}E_{ik}^{1-a_{ij}-n}E_{jl}E_{ik}^n\\ &=\sum_{n=0}^{1-a_{ij}}(-1)^n\begin{bmatrix}1-a_{ij}\\ n\end{bmatrix}F_{ik}^{1-a_{ij}-n}F_{jl}F_{ik}^n=0\qquad\text{if } i\in I^{re}\text{ and } i\neq j.\end{aligned} \tag{5.9}$$

The algebra $\mathbf{U}_v$ is equipped with a Hopf algebra structure as follows (see [1, 12]):

$$\Delta(K_i)=K_i\otimes K_i,$$
$$\Delta(E_{ik})=E_{ik}\otimes K_i^{-1}+1\otimes E_{ik},\qquad \Delta(F_{ik})=F_{ik}\otimes 1+K_i\otimes F_{ik},$$
$$\epsilon(K_i)=1,\quad \epsilon(E_{ik})=\epsilon(F_{ik})=0,$$
$$S(K_i)=K_i^{-1},\quad S(E_{ik})=-E_{ik}K_i,\quad S(F_{ik})=-K_i^{-1}F_{ik}.$$

We have an involution $\tau:\mathbf{U}_v\longrightarrow\mathbf{U}_v$ defined by $E_{ik}\mapsto F_{ik}$, $F_{ik}\mapsto E_{ik}$ and $K_i\mapsto K_i^{-1}$ for $i\in I$ and $k=1,\ldots,m_i$. We denote by $\mathbf{U}_v^0$ the subalgebra generated by $K_i^{\pm1}$, and by $\mathbf{U}_v^+$ (resp. $\mathbf{U}_v^-$) the subalgebra generated by E_{ik} (resp. F_{ik}). Similarly, we define $\mathbf{U}_v^{\geq 0}$ (resp. $\mathbf{U}_v^{\leq 0}$) to be the subalgebra generated by $K_i^{\pm1}$ and E_{ik} (resp. $K_i^{\pm1}$ and F_{ik}) for $i\in I$ and $k=1,\ldots,m_i$. Then

$$\mathbf{U}_v^0\cong\mathbb{C}[K_i^{\pm1}]_{i\in I},$$

and the involution τ identifies $\mathbf{U}_v^+$ with $\mathbf{U}_v^-$.

The following result provides a triangular decomposition for $\mathbf{U}_v$.

Proposition 5.9 ([1]). *The multiplication maps*

$$\mathbf{U}_v^0 \otimes_{\mathbb{C}} \mathbf{U}_v^+ \longrightarrow \mathbf{U}_v^{\geq 0}, \qquad \mathbf{U}_v^0 \otimes_{\mathbb{C}} \mathbf{U}_v^- \longrightarrow \mathbf{U}_v^{\leq 0},$$

and

$$\mathbf{U}_v^+ \otimes_{\mathbb{C}} \mathbf{U}_v^0 \otimes_{\mathbb{C}} \mathbf{U}_v^- \longrightarrow \mathbf{U}_v$$

are isomorphisms of vector spaces.

5.5. **Embedding of $\mathbf{U}_v^{\geq 0}$ into a Hall algebra.** Let $A = (a_{ij})_{i,j \in I}$ be a symmetric Borcherds–Cartan matrix such that each row has only finitely many nonzero entries and $a_{ii} \neq 0$ for any $i \in I$. Fix a locally finite quiver $\mathcal{Q}$ associated to A satisfying conditions (A) and (B): each vertex i has $1 - a_{ii}/2$ loops, and two distinct vertices i and j are connected with $-a_{ij}$ arrows for $i \neq j$. Then, since $a_{ii} \neq 0$ for any $i \in I$, the condition (B) is satisfied, and we can always choose an orientation for $\mathcal{Q}$ so that (A) is satisfied.

If $i \in I^{re}$, then there exists a unique simple object $S_i \in \mathrm{Rep}_{\Bbbk}(\mathcal{Q})$ supported at i. On the other hand, if $i \in I^{im}$ then the set of simple objects supported at i is in bijection with $\Bbbk^{c_i}$: if $\sigma_1, \dots, \sigma_{c_i}$ denote the simple loops at i then to $\underline{\lambda} = (\lambda_1, \dots, \lambda_{c_i}) \in \Bbbk^{c_i}$ corresponds the simple module $S_{i,\underline{\lambda}} = (V_j, x_\sigma)_{j \in I, \sigma \in \Omega}$ with $\dim_{\Bbbk} V_j = \delta_{ij}$ and $x_{\sigma_k} = \lambda_k \cdot \mathrm{id}$ for $k = 1, \dots, c_i$.

Let us now assume that the charge $\mathbf{m} = (m_i)_{i \in I}$ satisfies

$$m_i \leq |\Bbbk^{c_i}| = q^{c_i} \qquad \text{for each } i \in I.$$

We choose $\underline{\lambda}^{(l)} \in \Bbbk^{c_i}$ for $l = 1, \dots, m_i$ in such a way that $\underline{\lambda}^{(l)} \neq \underline{\lambda}^{(l')}$ for $l \neq l'$. Then we set $S_{il} = S_{i,\underline{\lambda}^{(l)}}$ for $i \in I^{im}$ and $l = 1, \dots, m_i$, and simply set $S_{i1} = S_i$ for $i \in I^{re}$. Since S_{il} have the same projectives in the standard resolution (5.2) for all $l = 1, \dots, m_i$, they define a unique class in $K(\mathcal{R})$. We will denote this unique class by $\hat{S}_i$ for any $i \in I$.

The following lemma will be used in the proof of Theorem 5.11.

Lemma 5.10. *Let $\bar{K}(\mathcal{A})$ denote the image of the map $K(\mathcal{A}) \to K(\mathcal{R})$ defined in* (2.7)*. Then the map $\mathbf{dim} : \bar{K}(\mathcal{A}) \to \mathbb{Z}^{\oplus I}$ is well-defined and is an isomorphism. That is,*

$$\mathbb{Z}^{\oplus I} \simeq \bar{K}(\mathcal{A}) \hookrightarrow K(\mathcal{R}).$$

Proof. The map $\mathbf{dim} : \bar{K}(\mathcal{A}) \to \mathbb{Z}^{\oplus I}$ is well-defined by condition (e) which is verified in Proposition 5.7. Clearly, the map is surjective. As noted already, there is a unique class $\hat{S}_i$ which represents all simple modules S_{il} in $\mathcal{A}$ for each $i \in I$. Thus the map is injective. □

The following theorem, due to Kang and Schiffmann, is an extension of well-known results of Ringel [20] and Green [9] from the case of a finite quiver without loops to a locally finite quiver with loops:

Theorem 5.11 ([13]). *Suppose $\Bbbk = \mathbb{F}_q$ is such that $|\Bbbk^{c_i}| \geq m_i$ for all $i \in I$. Then there are injective homomorphisms of algebras*

$$\mathbf{U}_v^+ \longrightarrow \mathcal{H}_v(\mathcal{A}) \qquad \textit{and} \qquad \mathbf{U}_v^{\geq 0} \hookrightarrow \widetilde{\mathcal{H}_v}(\mathcal{R})$$

defined on generators by $K_i^{\pm 1} \mapsto K_{\hat{S}_i}^{\pm 1}$ for $i \in I$,

$$E_{il} \mapsto [S_{il}] \cdot (q-1)^{-1} \qquad \textit{for } i \in I,$$

where S_{il} and $\hat{S}_i$ are defined right before Lemma 5.10.

Proof. Let $\widetilde{\mathcal{H}_{v,\Bbbk}}(\mathcal{Q})$ be the extended, twisted Hall algebra defined in [13]. It is shown in [13] that

$$\mathbf{U}_v^+ \hookrightarrow \mathcal{H}_v(\mathcal{A}) \qquad \text{and} \qquad \mathbf{U}_v^{\geq 0} \hookrightarrow \widetilde{\mathcal{H}_{v,\Bbbk}}(\mathcal{Q}).$$

Thus we have only to check that there exists an injective algebra homomorphism $\widetilde{\mathcal{H}_{v,\Bbbk}}(\mathcal{Q}) \hookrightarrow \widetilde{\mathcal{H}_v}(\mathcal{R})$. The only difference between the two algebras is that, while $\widetilde{\mathcal{H}_v}(\mathcal{R})$ is extended by $K(\mathcal{R})$, the algebra $\widetilde{\mathcal{H}_{v,\Bbbk}}(\mathcal{Q})$ is extended by $\mathbb{Z}^{\oplus I}$. Thus the embedding of $\widetilde{\mathcal{H}_{v,\Bbbk}}(\mathcal{Q})$ into $\widetilde{\mathcal{H}_v}(\mathcal{R})$ follows from Lemma 5.10. □

5.6. **Embedding of $\mathbf{U}_v$ into $\mathcal{DH}_{\mathsf{red}}(\mathcal{R})$.** We keep the notations in the previous subsection. In particular, the matrix A is a Borcherds–Cartan Matrix and $\mathcal{Q}$ is a fixed quiver corresponding to A. Suppose that $\mathbf{U}_v$ is the quantum group of the generalized Kac-Moody algebra associated with A.

Now we state and prove the main result of this paper.

Theorem 5.12. *There is an injective homomorphism of algebra*

$$\Xi\colon \mathbf{U}_v \hookrightarrow \mathcal{DH}_{\mathsf{red}}(\mathcal{R}),$$

defined on generators by

$$\Xi(E_{il}) = (q-1)^{-1} \cdot E_{S_{il}}, \qquad \Xi(F_{il}) = (-t)\cdot(q-1)^{-1} \cdot F_{S_{il}},$$
$$\Xi(K_i) = K_{\hat{S}_i}, \qquad \Xi(K_i^{-1}) = K^*_{\hat{S}_i},$$

where $l = 1, 2, \dots, m_i$ and $\hat{S}_i$ is the unique class representing S_{il} in $K(\mathcal{R})$ for each $i \in I$.

Proof. We have a commutative diagram of linear maps

$$\begin{tikzcd} \mathbf{U}_v^+ \otimes \mathbf{U}_v^0 \otimes \mathbf{U}_v^- \arrow[r, hook, "\Theta"] \arrow[d] & \mathcal{H}_v(\mathcal{A}) \otimes \mathbb{C}[K(\mathcal{R})] \otimes \mathcal{H}_v(\mathcal{A}) \arrow[d] \\ \mathbf{U}_v \arrow[r, hook, "\Xi"] & \mathcal{DH}_{\mathsf{red}}(\mathcal{R}) \end{tikzcd}$$

where the vertical arrows are the isomorphisms described in Propositions 4.9 and 5.9, respectively, and the homomorphism Θ is constructed out of the homomorphisms of Theorem 5.11. The map Ξ is a well-defined algebra homomorphism by Theorem 5.11 and by Lemmas 4.10, 4.11 and Corollary 4.7, which show that the generators satisfy the defining relations of the quantum group. It is clear from this diagram that Ξ is injective. □

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